1 Lecture II

Reminder. $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$. We would like to study representation $(\Pi, G, F(y))$, where $y = \Gamma\backslash G$.

We fixed a parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$ and consider a space $Z = \Gamma_p\backslash G$. The natural projection $pr : Z \to y$ identifies $F(y)$ with a G-invariant subspace of $F(Z)$.

1. For every $\lambda \in \mathbb{R}$ and $T > 0$ we introduced a seminorm $\rho_{\omega_\lambda,T}$ on the space $F(Z)$. When $T \ll 1$ this defines a norm on the subspace $F(y)$ and up to equivalence this norm does not depend on T.

Let us analyse properties of operators $\Pi(\alpha)$ for $\alpha \in \mathcal{H}$ with respect to this norm. Clearly operator $\Pi(\alpha)$ extends to the space $F(Z)$, but since the Siegel domain Z_T is not G-invariant the operator $\Pi(\alpha)$ in $F(Z)$ is not continuous with respect to the seminorm $\rho_{\omega\lambda,T}$.

We will patch this in the following way. Choose a compact ball $B \subset G$ which contains $supp(\alpha)$; we assume that B is symmetric, i.e. $B^{-1} = B$. Choose $T' < T$ such that $Z_T B \subset Z_{T'}$. Then clearly the operator $\Pi(\alpha)$ is continuous as an operator

$$(F(2), \rho_{\omega,T'}) \to (F(Z), \rho_{\omega,T}) \ .$$

Since on the subspace of interest $F(y)$ the norms $\rho_{\omega,T}$ and $\omega_{\omega,T'}$ are equivalent this is enough for our purposes.

2. The space Z is fibered by compact orbits of the group U acting on the left. This defines on the space $F(Z)$ the *cuspidal* structure, which is a canonical decomposition

$$(*) \qquad F(Z) = F_{cusp}(Z) + F_{const}(Z) \ .$$

The projector $\pi_p : F(Z) \to F_{const}(Z) = F(y_p)$ is given by $\pi_p(f)(Z) = \int_{u\in\Gamma_{U_p}\backslash U_p}(uZ)du$, which is essentially a constant term of f.

The decomposition $(*)$ is orthogonal with respect to every seminorm $\rho_{\omega,T}$. The first basic observation is that if a cuspidal function of $f \in F_{cusp}(X)$ is of uniform moderate growth then it is rapidly decreasing. Let us formulate the precise estimate which shows this. Fix a basis $\{\xi_i\}$ of the Lie algebra $\mathfrak{g}$.

Basic estimate. For a cuspidal function for Z we have an estimate

$$(*) \qquad \rho_{\omega_{\lambda+1}T}(f)^2 \leq C \circ \sum_i \rho_{\omega_\lambda,T}(\xi_i f)^2 \ .$$

3. Since the functions $\xi_i f$ are also cuspidal we can repeat this trick many times provided that f is smooth enough.

We can formalize it in the following way.

For every $k \in \mathbb{N}$ define a seminorm $\rho^k_{\omega,T}$ on $F(Z)$ by $\rho^k_{\omega,T}(f)^2 = \Sigma\rho_{\omega,T}(df)^2$, where the sum is over all monomials d in ξ_i of degree $\leq k$.

This seminorm measures the growth of f together with its derivatives up to order k.

Corollary of basic estimate. Let $f \in F(Z)$ be a cuspidal function. Then

$$\rho_{\omega_{\lambda+k,T}}(f) \leq C(k) \cdot \rho_{\omega,T}(f) \ .$$

In other words for cuspidal functions we can trade off smoothness for decay.

Corollary. *Suppose $\rho^k_{\omega,T}(f) < \infty$ for every k and f is cuspidal. Then $\rho_{\omega_\lambda,T}(f) < \infty$ for every λ.*

4. By definition the cuspidal part of the space $F(y)$ coincides with $F(y) \cap F_{cusp}(Z)$. We do not know whether this space is not zero. But we have seen that any smooth function in this space is rapidly decreasing.

Proposition. *Let $\mathcal{A}_{cusp}(y) \subset \mathcal{A}(y)$ be the subspace of cuspidal automorphic forms. Then*

(i) Every cusp form $f \in \mathcal{A}_{cusp}(y)$ is rapidly decreasing.

(ii) There exists a canonical decomposition $A(y) = A_{cusp}(y) \oplus \mathcal{A}^\perp_{cusp}$, where

$$\mathcal{A}^\perp_{cusp} = \{f \in \mathcal{A}(y) \ , \ f \text{ is orthogonal to } \ \mathcal{A}_{cusp}(y)\} \ .$$

Proof.

(i) Let $f \in \mathcal{A}_{cusp}(y)$. Then by Karish-Chandra's result we can find $\alpha \in \mathcal{H}$ s.t. $\Pi(\alpha)f = f$.

It is easy to see that $\rho^k_{\omega,T}(\Pi(\alpha)f) \leq \rho_{\omega,T}(f) \cdot \|\alpha\|_k$, where $\|\alpha\|_k$ is a C^k-norm of α. Thus the function f is of u.m.g. and hence rapidly decreasing.

(ii) Let us construct a projector

$$pr_{cusp} : \mathcal{A}(y) \to \mathcal{A}_{cusp}(y)\ .$$

Write $f_0 \in \mathcal{A}(y)$ as $\Pi(\alpha)f_0$ and consider the linear functional on the space $L^2_{cusp}(y)$ given by

$$f \mapsto \langle f_0, \Pi(\alpha^+)f\rangle\ .$$

Formally we can rewrite this as

$$\langle f_0, \Pi(\alpha^+)f\rangle = \langle \Pi(\alpha)f_0, f\rangle = \langle f_0, f\rangle\ .$$

But in fact the expression $\langle f_0, \Pi(\alpha^+)f\rangle$ is defined for all $f \in L^2_{cusp}(y)$ and is continuous in f. This follows from the fact that

$$\rho_{\omega_k}(\Pi(\alpha^+)f) \leq \rho^k_{\omega_0}(\Pi(\alpha^+)f) \leq C \cdot \rho_{\omega_0}(f) = C \cdot \|f\|_{L^2}\ .$$

Thus we can find a vector $h \in L^2_{cusp}(y) s.t. \langle h, f\rangle \equiv \langle f_0, \Pi(\alpha^+)f\rangle$.

This gives us the projection proof $\mathcal{A}(y)$ to $L^2_{cus[}(y)$. Clearly this cusp is well defined and commutes with actions of k and $z(G)$ (even of $U(\mathfrak{g})$). In particular, $pr(\mathcal{A}(y)) \subset \mathcal{A}_{cusp}(y)$. This is a required projector $pr_{cusp} : \mathcal{A}(y) \to \mathcal{A}_{cusp}(y)$ which leads to a direct sum decomposition.

5. Proof for the basic estimate. Let us assume that our basis ξ_i of $\mathfrak{g}$ is orthonormal w.r.t. some K invariant metric on $\mathfrak{g}$. We use this metric to define the left invariant Riemannian metric on G and hence Riemannian metric on Z. Then our inequality $(*)$ can be written as

$$\int_{Z_T} \omega_{\lambda+1}^2 f^2 d\mu \leq C \cdot \int \omega_x^2 \cdot (grad\ f)^2 d\mu\ .$$

5.(i) Now let us use the lemma from elementary calculus.

Lemma. *Consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with natural parameter a (and hence metric). Then for any smooth function h on S^1 we have*

$$\int_{S'} |h|^2 du - |\int h du|^2 \leq \int |grad h|^2 du\ .$$

This lemma has an immediate corollary.

Corollary. *Let $\nu : S^1 \to M$ be a smooth morphism of S^1 into some Riemannian manifold M. Suppose that the Lifschitz constant of ν is $\leq \nu$; in other words at every point $x \in S^1$ the tangent morphism $d\nu : T_x(S^1) \to T_{\nu(x)}(M)$ has norm $\leq \nu$. Then for every smooth function from M we have*

$$\int \nu^*|f|^2 du - |\int \nu^* f du|^2 \leq v^2 \cdot \int \nu^* |grad\ f|^2 du\ .$$

Indeed l.h.s. is bounded by $\int |grad(\nu^* f)|^2 du$, and pointwise we have an estimate

$$|grad(\nu^* f)^2 \leq r^2 \cdot \nu^*(|grad f|^2)\ .$$

This proves the corollary.

5(ii). now let us come back to the basic inequality.

Let $D = A \cdot K$. for every point $d \in D$ consider a morphism $\nu_d : S^1 = \Gamma_U \backslash U \to Z$ given by $u \mapsto ud$.

Then we can write the integral $\int_{Z_T} \omega^2 |f|^2 d\mu$ as an integral over D_T of $\int \nu_d^*(|f|^2) du$ with respect to some measure ν on D.

Similarly we can rewrite the integral $\int_{2T} \omega^2 |grad\ f|^2 d\mu$. Thus the basic inequality is reduced to the following estimate for every d

$$\int \nu_d^* |f|^2 du \leq (\int \nu_d^* |grad\ f|^2 du) a(d)^{-2}\ .$$

But this follows from Corollary 5(i) and the observation that ν_d has Lipschitz constant $\leq a(d)^{-1}$.

It is very clear from a geometric picture of the norm structure of the space $\widetilde{Z} = Z/K$.

Q.E.D.

6. Several more corollaries of the basic estimate.

Proposition *Let $f \in \mathcal{A}(y)$ an automorphic form. Consider $C(f) \in \mathcal{A}(y_p)$ and realize both f and $C(f)$ as a functions on Z¿ Then $\widetilde{f} = f - C(f)$ is rapidly decreasing, i.e.*

$$\rho^k_{\omega_\lambda, T}(\widetilde{f}) < \infty \quad \text{for every} \quad \lambda, k, T \ .$$

This was essentially proven in Proposition 4.

Warning. The function $\widetilde{f}$ does not usually lie in the space $F(y)$, so it has nothing to do with a cuspidal part of f described in Proposition 4(ii).

7. Proposition. *Fix a function $\alpha \in \mathcal{H}$ and consider an operator $\Pi(\alpha)$: $F(Z) \to F(Z)$. Choose $T' < T$ such that the operator $\Pi(\alpha)$ is continuous from $(F(Z), \rho_{\omega,T'})$ to $(F(Z), \rho_{\omega,T})$. Then on the subspace of cuspidal functions this operator is Hilbert-Schidt.*

Proof. I will describe a more sophisticated proof than what is done usually, but in my opinion it is more understandable.

Step 1. $\Pi(\alpha)$ is a continuous operator from $F(Z)_{cusp,\rho_{T',\omega}}$ to $F(Z)_{cusp,\rho^k_{T,\omega}}$ for any k.

This is true since all derivatives can be shifted to α.

Step 2. The identity operator $F(Z)_{cusp}\rho^k_{\omega_\lambda,T'} \to F(Z)_{cusp}, \rho^{k-e}_{\omega_{\lambda+e}}$ is a continuous operator.

Step 3. Proposition. Identity operator

$$F(Z), \rho^3_{\omega_{\lambda+2},T'} \to F(Z), \rho_{\omega_\lambda,T}$$

is a Hilbert-Schmidt operator.

In other words, if we sufficiently weaken the bounds for smoothness and decay of functions on Z, then the identity operator is a Hilbert-Schmidt operator.

This is a special case of a general statement about any Lie group. We will prove this proposition in an appendix.

8. Corollary. *Fix an ideal $I \subset z(G)$ and a subset $\sigma \in Irrk$ as above. Then the space $\mathcal{A}_\sigma^I(y)$ is finite dimensional and the dimension of this space can be effectively bounded.*

Proof. Consider the constant term operator $C : \mathcal{A}(y) \to \mathcal{A}(y_p)$. Then the image $C(\mathcal{A}_\sigma^I(y))$ lies in the subspace $\mathcal{A}_\sigma^I(y_p)$ which dimension is very easy to bound since y_p is a very simple space.

Thus it is enough to bound the dimension of $KerC \cap A_\sigma^I(y) = A_{\sigma,cusp}^{\pm}(y)$. By Harish-Chandra lemma there exists $\alpha \in \mathcal{H}$ such that $\Pi(\alpha) = Id$ on the space $\mathcal{A}_\sigma^I$.

Consider the operator $\Pi(\alpha)$ on $L_{cusp}^2(y)$. Then Proposition 7 easily implies that this is a Hilbert-Schmidt operator and moreover, it allows us to explicitly bound its Hilbert-Schmidt norm, $\|\Pi(\alpha)\|_{HS}^2 \leq D$. Since on the subspace $\mathcal{A}_{\sigma,cusp}^I \subset L_{cusp}^2(y)$ this operator is an identity, we get that $\dim \mathcal{A}_{\sigma,cusp}^I \leq D$.