Meromorphic Continuation of Eisenstein Series

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1 Lecture I

Case of $G = SL(2, \mathbb{R})$.

1. Consider the group $G = SL(2, \mathbb{R})$. Set $\mathfrak{g} = Lie(G)$ – Lie algebra of G. $U(\mathfrak{g})$ – universal enveloping algebra $z(G)$:= center of $U(\mathfrak{g})$.

$z(G)$ is the algebra of bi-invariant differential operators on G.

In case of $G = SL(2, \mathbb{R})$ the algebra $z(\mathfrak{g})$ is the polynomial algebra $\mathbb{C}[C]$, where c is the Casimir operator (some operator of order 2).

Fix a maximal compact subgroup $K = SO(2)$ in G. Sometimes it is convenient to consider the quotient space G/K. This is a classical upper half plane

$$H = \{2 = x + iy \in \mathbb{C} \mid y > 0\}$$

action of G on H is given by fractional linear transformations, and K is the stabilizer of the point $x_0 = i$.

2. Fix a discrete subgroup $\Gamma \subset G$ (we mostly are interested in case $\Gamma = SL(2, \mathbb{Z})$). Consider the space $y = \Gamma \backslash G$. This space has canonical right action of G. Hence we get a representation.

$(\prod, G, F(y))$ where $F(y)$ is some space of functions on y.

Basic definition

An automorphic form is a function $f \in F(y)$ which satisfies

(i) f is K-finite

(ii) f is $z(G)$-finite

(iii) f has moderate growth ($= f$ is tempered). We will discuss the precise definition later.

3. Let $\mathcal{A}(y)$ be the space of automorphic forms. By definition it is a union of subspaces of the following sort.

Fix an ideal $I \subset z(G)$ of finite codimension. Fix a finite collection σ of irreducible representations of k. Set $\mathcal{A}_\sigma^I = \{f \in F(y) \mid If = 0, f \in F(y)_\sigma\}$, where $F((y)_\sigma$ is the space of a function which under the action of k gives only irreducible representations in σ.

One of the results we would like to explain is

Theorem. *for any I, σ the space $\mathcal{A}_\sigma^I$ is finite dimensional.*

Remark. Why we need temperedness condition. For example, consider $\Gamma = SL(2,\mathbb{Z})$ and consider a harmonic Γ-invariant function f on H. Then $cf = 0$ since on the space $F(H)$ C acts as $y^2(\partial_x^2 + \partial_y^2)$. There are too many of such f if we do not compose the temperedness condition. For example for every λ the function $f_\lambda = \exp(\lambda, j)$, where j is j-invariant, is harmonic on $\Gamma \backslash H$.

4. Let f be a $z(G)$ – finite function. Then by a standard argument we can decompose $f = \Sigma$ to where the sum is taken over characters θ of the algebra $z(G)$ and f_θ is annihilated by some power of $\ker(\theta : z(G) \to \mathbb{C})$. Thus we can canonically decompose the space $\mathcal{A}(y)$ into a direct sum $\mathcal{A}^\theta(y)$.

Similarly, we have decomposition $\mathcal{A}(y) = \bigoplus_{\rho \in I_{rr}k} \mathcal{A}_\rho(y)$.

5. Consider $\Gamma = SL(2,\mathbb{Z})$. Fix a standard parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \subset G$. Set U_p =unipotent radical of P, $U_p = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\}$. Set

$L_p = p/U_p$, $L_p \approx \mathbb{R}^*$ (L_p is called Levi group). Set $\Gamma_p = \Gamma \cap p$, $\Gamma_{U_p} = \Gamma \cap U_p$, Γ_{L_p} =Image of Γ_p in L_p.

We can study automorphic forms on the space y comparing them with automorphic forms on the space $y_p = \Gamma_p U_p \backslash G$. G acts on y_p on the right and preserves a measure μ. Moreover, if we fix a measure on y we get canonical measure on y_p. Thus G and hence $z(\mathfrak{g})$ acts on the space $F(y_p)$.

Definition. A function f on y_p is called automorphic if it is k-finite, $z(\mathfrak{g})$-finite and of moderate growth. $\mathcal{A}(y_p)$ the space of automorphic functions on y_p. Space $F(y_p)$ has additional symmetries. Namely consider the basic affine space $Aff_p = U_p \backslash G$. Then it has right action of G, but also left action of L_p. Note that the action of G on $F(Aff_p)$ preserves L^2-structure given by a G-invariant measure on μ on Aff_p, but the action of L_p multiplies μ by a character. So we would always modify the action of L_p on the space $F(y_p)$ in such a way that it preserves L^2-structure, action of L_p defines the action of the algebra $z(L_p)$.

Lemma. (Kavish-Chandra). *$z(L_p) \supset z(G)$ and $z(L_p)$ is an f-generated $z(G)$ module.*

We will identify $y_p = \Gamma_{L_p} \backslash Aff_p$. Thus on the space $F(y_p)$ we have an additional action of $z(L_p)$. Clearly that $z(G)$-finite functions are also $z(L_p)$-finite, but $z(L_p)$ has more characters than $z(G)$. Thus for $\mathcal{A}(y_p)$ we have finer decomposition $\mathcal{A}(y_p) = \Sigma \mathcal{A}^x$, f-characters of $z(L_p)$.

Explicit description $Aff_p = \mathbb{R}^2 \backslash 0$. Action of L_p on $F(Aff_p)$ is given by homotheties (twisted). Generator of L_p acts as an operator $\delta = -(x_1 \partial_1 + x_2 \partial_2 + 1)$. We have $\delta^2 = C + 1$.

$\mathcal{A}^s(y_p)$ consists of homogeneous k-finite functions of degree $-s-1$ $y_p = \{\mathbb{R}^2 \backslash 0\}/\{\pm 1\}$.

6. Operators E and C.

Spaces $F(y)$ and $F(y_p)$ are connected via canonical operators

$$E : F(y_p) \longrightarrow F(y) \quad \text{(Eisenstein operator)}$$

$$C : F(y) \longrightarrow F(y_p) \quad \text{(constant term operator)} \ .$$

In order to define them notice that the choice of a measure on y canonically defines a measure on y_p, since the measure on U_p can be normalized by the condition $\mu(\Gamma_{U_p} \backslash U_p) = 1$.

Set $Z_p = \Gamma_p \backslash G$ and consider a double fibration $y_p \leftarrow Z \rightarrow y$. This double fibration defines operators E and C. Explicitly

$$E(\varphi) = \sum_{\gamma \in \Gamma / \Gamma_p} \gamma(\varphi)$$

$$C(f) = \int_{u \in U_p / \Gamma_{U_p}} uf \ .$$

These operators are partially defined since the series for E does not always converge. But where they are defined they are adjoint.

The subspace of $F(y)$ annihilated by the constant term operator C is called the space of cuspidal functions and denoted $F_{cusp}(y)$. Note that the operator C maps $\mathcal{A}(y)$ into $\mathcal{A}(y_p)$. Its kernel $\mathcal{A}_{cusp}(y)$ is called the space of *cuspidal automorphic functions* (or *cusp forms*). One of the important results we will prove is the following

Lemma. *Any cusp form $f \in \mathcal{A}_{cusp}(y)$ is rapidly decreasing.*

Corollary. *There exists a canonical decomposition $\mathcal{A}(y)\mathcal{A}_{cusp}(y) \oplus \mathcal{A}_{Eis}(y)$, where $\mathcal{A}_{Eis} = \mathcal{A}_{cusp}^{\perp}$.*

6'. Operators E and C could be also described geometrically.

Let us identify $U \backslash G$ with the space $Aff_p = \mathbb{R}^2 \backslash 0$. It has a canonical G-invariant measure which is the standard volume on $\mathbb{R}^2$. In our case $y_p = Aff_p / \{\pm\}$.

The space y we can interpret as the space of all immodular lattices $\Lambda \subset \mathbb{R}^2$.

With these interpretations we have

$$E(\varphi)(\Lambda) = \sum_{u \in \Lambda / \pm 1, u-primitive} \varphi(u)$$

$$C(f)(u) = \int_{x \in \mathbb{R}/\mathbb{Z}} f(\Lambda(u, v_x) dx, \quad \text{where}$$

we fixed a vector $v \in \mathbb{R}^2$ such that the volume (u, v) equals 1, set for every $x \in \mathbb{R}$ $v_x = v + xu \in \mathbb{R}^2$ and denote by $\Lambda(u, v_x)$ the lattice generated by u and v_x.

7. Eisenstein operator E allows to produce automorphic forms on y from automorphic forms on much simpler space y_p.

Namely, start with an automorphic form $\varphi \in \mathcal{A}(y_p)$ and consider a family of automorphic forms on y_p given by $u_s = \varphi, a^s$, where $a : y_p \to \mathbb{R}^{+*}$ is defined by $a(y) = \|y\|^{-2}$.

We will prove the theorem.

Theorem. (i) *For $Res \gg 0$ the series $E(\varphi, s) = E(u_s)$ is absolutely convergent and is an automorphic form on y.*
(ii) *The function $s \mapsto E(\varphi, s) \in F(y)$ is holomorphic in s for $Res \gg 0$.*
(iii) *The function $E(s)$ has meromorphic continuation to the whole complex plane of parameter s.*

8. **Convolution operators.** Let $\mathcal{H} = c_c^{\infty}(G)$ be considered as an algebra with respect to convolution. Any representation (π, G, V) defines a representation of this algebra $\pi(\alpha) = \int \alpha(g)\pi(g)d\mu$ in the space V. This is a smoothing operator.

For general α the operator $\pi(\alpha)$ does not preserve the subspace of k-finite vectors. However, if we consider α in the subalgebra $\mathcal{H}^k \subset \mathcal{H}$ of $Ad(k)$-invariant functions the operator $\pi(\alpha)$ preserves each subspace V_σ, $\sigma \in IrrK$.

Proposition (Harish-Chandra). *Fix $I \subset z(G)$ and $\sigma \subset IrrK$ as above. Then for any neighborhood U of point $c \in G$ one can find a function $\alpha \in \mathcal{H}^K$, supported in U, such that $\pi(\alpha) = Id$ on the subspace V_σ^I.*

In particular, for all automorphic forms $f \in \mathcal{A}_\sigma^I$ we have $\Pi(\alpha)f = f$. In particular this implies that f is smooth of uniform moderate growth.

9. The main analytic difficulty in studying the space $F(y)$ is that the space y is not compact and thus we have to understand the structure of y

near ∞.

Let us try to see the structure of y by considering it as a metric space (metric comes from a left invariant metric on G) and "looking at it from afar", i.e. paying attention only to large scale structures.

On a large scale y is equivalent to the space $\widetilde{y} = y/k$. The space $\widetilde{y}$ is easier to understand since $\widetilde{y} = \Gamma\backslash H$. Replacing $\widetilde{y}$ by its standard fundamental domain in H we see that on a large scale it is equivalent to the segment $[1, \infty) \subset \mathbb{R}^{+*}$. Let me describe a method how to use these observations in order to describe the analytic structures on the space $F(y)$.

10. Consider the subgroup $\Gamma_p = \Gamma \cap P$ and set $Z = \Gamma_p\backslash G$. We have the natural projection $Z \to y$ and thus we can identify the space of functions $F(y)$ as a subspace of $F(Z)$. We will see that the space Z has relatively simple structure so that all the estimates in the space of functions $F(Z)$ are relatively simple. We will develop these estimates and then apply them to the complicated subspace $F(y) \subset F(Z)$.

Geometrically the space $\widetilde{Z} = Z/K$, which equals $\Gamma_p\backslash H = \Gamma_u\backslash H$ is a cylinder (quotient of H by integral translations in x coordinate). However, since the metric on H is given by $ds^2 = \frac{dx^2+dy^2}{y^2}$, it is better to think about $\widetilde{Z}$ as a horn;

PICTURE FOR $\widetilde{Z}$

We have seen that for the description of the space $\widetilde{y}$ we need only the narrow part of this horn. This motivates the following definition. Consider natural projection $a : Z \to \mathbb{R}_*^+$ given by $a(z) = y$ – coordinate of the point Z module K, considered as a point of H.

Definition. For every $T > 0$ consider domain $Z_T = \{z \in Z \mid a(z) \geq T\}$.

We call these domains Z_T *Siegel* domains, though formally Siegel considered corresponding domains in G, not in Z.

Reduction theory. (i) For every T the projection $pr : Z_T \to y$ has finite fibers of size bounded by $C(T)$.
(ii) If $T \ll 1$, then $pr : Z_T \to y$ is onto.

Thus given a function f on y we can completely describe its analytic properties in terms of analytic properties of its lifting $f^* \in F(Z)$ restricted to a Siegel domain Z_T with some fixed $T \ll 1$.

11. For every $\lambda \in \mathbb{R}$ consider a weight function $\omega = \omega_\lambda$ on Z defined by $\omega_\lambda(z) = a(z)^\lambda$. Now for a given weight function ω on Z and a number T defines a seminorm on the space $T(Z)$

$$\rho_{\omega,T}(f)^2 = \int_{Z_T} ||\omega f||^2 d\mu_Z$$

(roughly $\rho_{\omega T}$ measures the size of the function ωf on Z_T).

Reduction theory immediately implies

Proposition *Consider the seminorm $\rho_{\omega,T}$ on the subspace $F(y) \subset F(Z)$.*

(i) *Suppose $T \ll 1$. Then $\rho_{\omega,T}$ is a norm on $F(y)$.*

(ii) *Given T, T', both $\ll 1$, the norms $\rho_{\omega,T}$ and $\rho_{\omega,T'}$ are equivalent.*

Thus using the space Z we have defined on $F(y)$ a system of norms ρ_ω defined up to equivalence. We denote $L^2_\omega(y)$ the completion of the space $F(y)$ with respect to the norm ρ_ω. This gives us a decreasing family of subspaces $L^2_{\omega_\lambda}(y)$ in $F(y)$.

Definition. Let f be a function on y.

(i) We say that f is of *moderate growth* (or that f is a *tempered* function on y) if $f \in L^2_{\omega_\lambda}(y)$ for some weight ω_λ(i.e. $\rho_{\omega_\lambda,T}(f) < \infty$).

(ii) We say that f is *rapidly decreasing* at infinity if $f \in L^2_\omega(y)$ for all λ (i.e. $\rho_{\omega_\lambda,T}(f) < \infty$) for every λ).