NOTES ON 4-MANIFOLDS

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Abstract. These are notes on 4-manifolds, with a focus on the theory of smooth, simply-connected manifolds, and the (elementary) use of gauge theory. These notes are based on a graduate course co-taught with Benson Farb at the University of Chicago in Winter 2018, but represent only the part of the course taught by Danny Calegari.

Contents

1. Topological 4-manifolds

We say that a manifold is *closed* if it is compact without boundary. We usually assume our manifolds are connected unless we explicitly say otherwise.

1.1. Examples of 4-manifolds.

Example 1.1 (4-sphere). The 4-sphere is the simplest closed 4-manifold, being the 1-point compactification of $\mathbb{R}^4$. It does not admit a complex structure; this can most easily be shown using characteristic classes (see Example 1.6). However, it *does* admit a *quaternionic structure* in its realization as the *quaternionic projective line*.

Example 1.2 (Complex and algebraic surfaces). A 2-dimensional complex manifold (usually called a *surface*) is a 4-dimensional real manifold. A complex surface is *algebraic* if it arises as the subset of $\mathbb{CP}^n$ consisting of the common zeros of a collection of homogeneous polynomials in $n+1$ variables. An algebraic surface is necessarily *Kähler*; i.e. it has a metric for which parallel transport preserves both the metric and the complex structure on tangent spaces. This has many consequences for the topology of the surface, for example one must have first Betti number b_1 even.

One example is $\mathbb{CP}^2$ itself. Note that complex manifolds are canonically *oriented*. The same manifold with the opposite orientation is denoted $\overline{\mathbb{CP}}^2$, and is *not* a complex manifold (see Example 1.10).

Some complex manifolds are not algebraic; for example, the *Hopf surfaces*, obtained as a quotient of $\mathbb{C}^2 - 0$ by a cyclic group generated by $(z,w) \to (\lambda z, \lambda w)$ with $|\lambda| \neq 1$.

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Topologically, all such surfaces are diffeomorphic to $S^3 \times S^1$. They have $\pi_1 = \mathbb{Z}$ so that b_1 is odd, and are therefore not algebraic (or they would be Kähler).

Example 1.3 (K3). A generic quartic X in $\mathbb{CP}^3$ (for example, the zero locus of $x^4+y^4+z^4+w^4=0$) is nonsingular, and defines a complex surface known as a *K3 surface*. There is a *Veronese embedding* $\mathbb{CP}^3 \to \mathbb{CP}^{34}$, where the (projective) coordinates of the big projective space are the $\binom{7}{4} = 35$ monomials of degree 4 in the variables x, y, z, w (a monomial like x^2yw can be encoded as $**|*||*$). Under this embedding, the quartic becomes the intersection of $\mathbb{CP}^3$ with a hyperplane H. The *Lefschetz hyperplane theorem* says that $\pi_i(\mathbb{CP}^3 \cap H) \to \pi_i(\mathbb{CP}^3)$ and $H_i(\mathbb{CP}^3 \cap H) \to H_i(\mathbb{CP}^3)$ are isomorphisms for $i < 2$ and a surjection for $i = 2$, so X is simply-connected. We shall say more about the topology of X in the sequel.

See e.g. Griffiths and Harris [4] Chapter 0 for more on the Veronese embedding and the Lefschetz hyperplane theorem.

Example 1.4 (Hyperbolic manifolds). Suppose G is a semisimple Lie group, Γ is a lattice in G, and K is a compact subgroup of G of codimension 4. Then there is a torsion-free subgroup of Γ of finite index, and the quotient of G/K by this subgroup is a 4-manifold X which is locally symmetric.

For example, if $G = \mathrm{O}(4,1)$ (resp. $\mathrm{U}(2,1)$, $\mathrm{Sp}(1,1)$) and $K = \pm\mathrm{O}(4)$ (resp. $\mathrm{U}(2)\times\mathrm{U}(1)$, $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$), then X is a real (resp. complex, quaternionic) 4 (resp. 2, 1) manifold.

Lattices in $\mathrm{O}(4,1)$, $\mathrm{U}(2,1)$ and $\mathrm{Sp}(1,1)$ can be constructed from arithmetic. Other examples arise e.g. as subgroups of Coxeter groups. For instance, there is a regular 120 cell in real hyperbolic 4-space with all dihedral angles $\pi/2$, whose faces are regular right-angled 3-dimensional hyperbolic dodecahedra. The group generated by reflections in the side of this polyhedron is discrete and cocompact, and has many finite index subgroups which are torsion-free

Example 1.5 (Products and bundles). Products of low dimensional manifolds give examples of 4-manifolds. For example: $S^3 \times S^1$ which has fundamental group $\mathbb{Z}$; the 4-torus T^4 with fundamental group $\mathbb{Z}^4$; or products of surfaces.

One can also consider bundles.

1. Oriented circle bundles over a 3-manifold M are classified by elements of $H^2(M;\mathbb{Z})$.
2. There are exactly two oriented S^2 bundles over a closed oriented surface Σ, classified by elements of $H^2(\Sigma;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. The nontrivial bundle is obtained from the trivial one as follows: pick a disk $D \subset \Sigma$ and remove the interior of the product $D \times S^2$; then glue antipodal fibers of $\partial D \times S^2$ to each other by the 1-parameter family of diffeomorphisms coming from the nontrivial element of $\pi_1(\mathrm{SO}(3))$.
3. Σ_g bundles over Σ_h for $g > 1$ are classified by the monodromy homomorphism from $\pi_1(\Sigma_h)$ to the mapping class group of Σ_g.
4. A T^2 bundle over Σ is classified by the monodromy $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{Z})$ *and* the Euler class, which is a pair of integers. This reflects the fact that the identity component of $\mathrm{Diff}(T^2)$ is itself homotopy equivalent to T^2, so that its classifying space is a product $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$.
5. One often considers surface bundles over surfaces with finitely many *singular fibers*, usually of a restricted type. The most common example are *Lefschetz pencils*, such

as those that arise in algebraic or symplectic geometry. In this case, some of the fibers are *nodal curves*, where a simple curve on the (generic) fiber (the *vanishing cycle*) has pinched to a point. A loop in the base around the singular point exhibits monodromy: a Dehn twist around the vanishing cycle.

1.2. **Characteristic classes.** A basic reference for characteristic classes is Milnor and Stasheff [9].

1.2.1. *Stiefel–Whitney and Pontriagin classes.* For a closed, smooth 4-manifold W we have Stiefel–Whitney classes $w_i \in H^i(W;\mathbb{Z}/2\mathbb{Z})$ for $i = 1, \cdots, 4$ and Pontriagin class $p_1 \in H^4(W;\mathbb{Z})$. We have $w_1 = 0$ iff W is orientable, and for an orientable manifold, $w_2 = 0$ iff W admits a spin structure (one can take this as a definition for now). Naturally, if W is simply-connected then $w_1 = w_3 = 0$.

If W is orientable, there is also the *Euler class* $e \in H^4(W;\mathbb{Z})$ which satisfies $e([W]) = \chi(W)$, the Euler characteristic, and $e = w_4 \bmod 2$. Thus, for simply-connected W the relevant classes are w_2, e, p_1. Note further that for simply-connected W we must have $\chi(W) \geq 2$.

1.2.2. *Chern classes.* A smooth manifold W is said to be *almost complex* if there is a section J of the endomorphism bundle of TW with $J^2 = -\mathrm{id}$ fiberwise; equivalently, if TW can be given the structure of a 2-dimensional complex vector bundle. Any complex manifold is almost complex, but not conversely.

An almost complex structure on W gives rise to Chern classes $c_i \in H^{2i}(W;\mathbb{Z})$ for $i = 1, 2$. An almost complex manifold is necessarily orientable, and $c_2 = e$. Thus $c_2([W]) = \chi(W)$. Further, $c_1 = w_2 \bmod 2$.

Since for any bundle E we have $p_1(E) = -c_2(E \otimes \mathbb{C})$ by definition, if TW already has a complex structure, $TW_{\mathbb{C}} = TW \oplus \overline{TW}$ so that

$$1 - p_1 = 1 + c_1(TW_{\mathbb{C}}) + c_2(TW_{\mathbb{C}}) = (1 + c_1 + c_2) \cdot (1 - c_1 + c_2) = 1 - c_1^2 + 2c_2$$

Example 1.6 (The 4-sphere). The 4-sphere does not admit a complex structure (or even an almost-complex structure). For, if it did, we would have $p_1 = c_1^2 - 2c_2$, and therefore $p_1[S^4] = -2c_2[S_4] = -2\chi(S^4) = -4$. But the 4-sphere is a boundary, and therefore its Pontriagin numbers vanish, as follows from Stokes' Theorem (also see § 1.3.1).

Example 1.7 ($\mathbb{CP}^2$). The definition of Chern classes says that the tautological line bundle γ over $\mathbb{CP}^2$ has $c_1 = a$, the generator of H^2. If ω is the orthogonal complement of γ in a trivial $\mathbb{C}^3$ bundle over $\mathbb{CP}^2$ then $T\mathbb{CP}^2 = \mathrm{Hom}(\gamma, \omega)$. Thus $T\mathbb{CP}^2 \oplus \epsilon^1$ (where ϵ^1 is a trivial line bundle) is isomorphic to the sum of 3 copies of γ^*, so the total Chern class of $\mathbb{CP}^2$ is $(1 - a)^3 = 1 - 3a + 3a^2$. In other words, $c_1 = -3a$ and $c_2[\mathbb{CP}^2] = 3$. Thus $p_1[\mathbb{CP}^2] = 9 - 6 = 3$.

1.3. **Intersection forms.** If W is a closed, oriented 4-manifold, the *intersection pairing* on $H^2(W;\mathbb{Z})$ is the quadratic form defined by cup product:

$$\alpha \cdot \beta := (\alpha \cup \beta)([W])$$

where $[W] \in H_4(W, \mathbb{Z})$ is the fundamental class associated to the orientation. Note that this makes sense even if W is not connected. Changing the orientation produces a new oriented manifold which we denote by $\overline{W}$, and replaces the intersection pairing by its negative.

Here is a geometric way to see the cup product. For any closed, oriented, smooth 4-manifold W, a class $\alpha \in H^2(W;\mathbb{Z})$ is represented by the homotopy class of a map $f_\alpha : W \to \mathbb{CP}^2$ (by the fact that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$, and cellular approximation). Homotop f_α to be smooth, and transverse to a hyperplane $H \subset \mathbb{CP}^2$. Then $f_\alpha^{-1}(H)$ is a smooth oriented embedded surface in W representing a homology class A which is Poincaré dual to α. If α, β are cohomology classes associated to surfaces A, B in W, then we may isotop A and B to be transverse, and count the number of intersections with sign. Then

$$\alpha \cdot \beta = \sum_{A \cap B} \pm 1$$

The intersection pairing is symmetric, and vanishes on the torsion part of H^2. If we choose an integral basis for the free part of H^2, then the pairing is given by a symmetric matrix Q. Note that the matrix Q has integer entries; further, since Poincaré duality gives

$$H^2(W;\mathbb{Z}) = H_2(W;\mathbb{Z})$$

and the universal coefficient theorem gives

$$H^2(W;\mathbb{Z}) = \mathrm{Hom}(H_2(W;\mathbb{Z});\mathbb{Z}) \oplus \text{torsion}$$

it follows that the determinant of Q is ± 1. We express this by saying that the quadratic form is *symmetric*, *integral*, and *unimodular*.

Two integral forms are *equivalent* if one is taken to the other by an integral change of basis. This replaces Q by $A^T QA$ for some $A \in \mathrm{GL}(n,\mathbb{Z})$ where n is the rank of H^2.

1.3.1. *Signature.* Associated to the equivalence class of a symmetric integral unimodular form are three invariants:

(1) the *rank*;
(2) whether the form is *even* (i.e. $\alpha \cdot \alpha$ is even for all α) or *odd*; and
(3) the *signature*, which is equal to the number of positive eigenvalues minus the number of negative eigenvalues.

The signature is denoted σ, and is sometimes also called the *index*. If Q is the pairing on $H^2(W;\mathbb{Z})$, the rank is b_2, and the signature is $b_2^+ - b_2^-$, and are thefore the same mod 2.

Changing the orientation gives $Q(\overline{W}) = -Q(W)$ so $\sigma(\overline{W}) = -\sigma(W)$. Likewise, for any two 4-manifolds W, V we have $Q(W \sqcup V) = Q(W) \oplus Q(V)$ thus $\sigma(W \sqcup V) = \sigma(W) + \sigma(V)$.

Theorem 1.8 (Cobordism)**.** *Let W be smooth, closed, and oriented. Then $W = \partial M$ for some compact, smooth, oriented 5-manifold if and only if $\sigma(W) = 0$.*

Proof. We show that $W = \partial M$ implies $\sigma(W) = 0$. Poincaré–Lefschetz duality gives the following commutative diagram:

$$\begin{array}{ccccc} H_3(M,W) & \xrightarrow{\partial_*} & H_2(W) & \xrightarrow{i_*} & H_2(M) \\ & & \downarrow \cong & & \downarrow \cong \\ & & H^2(W) & \xrightarrow{\delta^*} & H^3(M,W) \end{array}$$

where i_* is induced by inclusion. The maps ∂_* and δ^* are adjoint with respect to the (nondegenerate) pairing of H^2 with H_2, and hence the ranks of $\operatorname{coker}\partial_*$ and $\ker\delta^*$ are equal. Hence the ranks of $H_2(W)/\operatorname{im}\partial_*$ and $\ker i_* = \operatorname{im}\partial_*$ are equal, and therefore the rank of $\ker i_*$ is *half* the rank of $H_2(W)$.

Now, this subspace is isotropic for the intersection pairing in homology; dually, there is a subspace of $H^2(W)$ of half dimension isotropic for Q. But if Q has p positive eigenvalues and q negative eigenvalues, the maximal dimension of an isotropic subspace is $\min(p, q)$. Thus $p = q$ and $\sigma(W) = p - q = 0$.

The converse is a special case of Thom's theory of cobordism, and is much harder; an elementary proof is given in Kirby [6]. ☐

Theorem 1.8 shows that σ is an *isomorphism* from the smooth oriented cobordism group Ω_4^{SO} to $\mathbb{Z}$. A proof of this fact mod torsion is given in [9], and this is enough to deduce the following:

Corollary 1.9 (Hirzebruch Signature Theorem)**.** *For W smooth and closed, $p_1[W] = 3\sigma(W)$.*

Proof. By Theorem 1.8, the disjoint union $W \sqcup \overline{V}$ is an oriented boundary if and only if $\sigma(W) = \sigma(V)$. Furthermore, since p_1 can be defined using the curvature of a connection, $p_1(W) = p_1(V)$ if $W \sqcup \overline{V}$ is an oriented boundary. Thus p_1 and σ are proportional, and we just need to compute one nontrivial example.

By Example 1.7 we have $p_1[\mathbb{CP}^2] = 3 = 3\sigma(\mathbb{CP}^2)$. ☐

Example 1.10 ($\overline{\mathbb{CP}}^2$). The manifold $\overline{\mathbb{CP}}^2$ (i.e. the complex projective plane with the 'wrong' orientation) is not almost-complex. For, we still have $c_2 = \chi = 3$. But $p_1 = 3\sigma = -3$, and therefore $-3 = c_1^2 - 6$ so $c_1^2 = 3$ which is absurd, because the hyperplane class squares to -1 (with the given orientation).

Example 1.11 (K3). Let X be the K3 surface; i.e. the generic quartic in $\mathbb{CP}^3$ (see Example 1.3). We compute its homology using Chern classes. We have already seen that X is simply-connected, so that $H_1 = H_3 = 0$ and $H_4 = \mathbb{Z}$.

If $a \in H^2(\mathbb{CP}^3; \mathbb{Z})$ is the generator, the total Chern class of $\mathbb{CP}^3$ is

$$1 + c_1(\mathbb{CP}^3) + c_2(\mathbb{CP}^3) + c_3(\mathbb{CP}^3) = (1 + a)^4 = 1 + 4a + 6a^2 + 4a^3$$

Let $x \in H^2(X; \mathbb{Z})$ be obtained by pulling back a under the embedding, so that $T\mathbb{CP}^3|_X$ has total Chern class $1 + 4x + 6x^2$. The tangent bundle $T\mathbb{CP}^3$ restricts to X as $TX \oplus \nu$ where ν is the normal bundle. Now, $c_1(\nu)$ is Poincaré dual to the self-intersection class of X; i.e. to the submanifold $X \cap X'$ where X' is another quartic in general position. Now, x is the pullback of a, which is Poincaré dual (in $\mathbb{CP}^3$) to a hyperplane H, so $c_1(\nu) = 4x$.

We therefore compute

$$1 + c_1(X) + c_2(X) = (1 + 4x + 6x^2)(1 + 4x)^{-1} = (1 + 4x + 6x^2)(1 - 4x + 16x^2) = 1 + 6x^2$$

so that X has $c_1 = 0$ and $c_2 = 6x^2$. Now, $x^2[X] = a^2[X] = 4$ so $\chi(X) = c_2[X] = 24$ and $H^2(X) = \mathbb{Z}^{22}$. Furthermore, $3\sigma(X) = p_1[X] = c_1^2[X] - 2c_2[X] = -48$ so $\sigma(X) = -16$ and $b_2^+ = 3$, $b_2^- = 19$.

1.3.2. *Indefinite forms.* A form is *indefinite* if it has both positive and negative eigenvalues.

The following theorem says that an indefinite form has an *isotropic* element; i.e. an element that squares to zero. Such an element can be taken to be primitive.

Theorem 1.12 (isotropic)**.** *Suppose Q is indefinite. Then there is a primitive x with $x \cdot x = 0$.*

Note that it suffices to show that the quadratic form coming from Q has nontrivial zeroes over the field of rationals. If the rank is < 5 this can be shown by an ad hoc argument. If the rank is at least 5, one shows first that any quadratic form has a zero in every p-adic completion $\mathbb{Q}_p$ (this is basically a counting argument in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$) and then applies the theorem of Hasse–Minkowski that a quadratic form which represents zero in every completion of $\mathbb{Q}$ represents zero already in $\mathbb{Q}$, so that Q represents zero over $\mathbb{Q}$ if and only if it represents zero over $\mathbb{R}$; i.e if and only if it is indefinite.

See Serre [12], Ch. V, Thm. 3 for a proof.

1.3.3. *Even and Odd forms.* The intersection form on $H^2(W;\mathbb{Z})$ is *even* if $\alpha \cdot \alpha$ is even for all $\alpha \in H^2(W;\mathbb{Z})$. Note that a form is even if and only if $\alpha \cdot \alpha$ is even for each α in a set of basis vectors; equivalently, if the matrix Q has even numbers down the diagonal.

Let us suppose that W is simply-connected, so that $H^2(W;\mathbb{Z}/2\mathbb{Z}) = H^2(W;\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$. The quadratic form $\alpha \to \alpha \cdot \alpha$ becomes *linear* when reduced mod 2, and therefore there is some $\omega \in H^2(W;\mathbb{Z})$ so that $\omega \cdot \alpha = \alpha \cdot \alpha$ mod 2 for all $\alpha \in H^2(W;\mathbb{Z})$. We call such an ω *characteristic*, and observe that its mod 2 reduction is unique. Thus, any other ω' is of the form $\omega' = \omega + 2x$ and therefore

$$\omega' \cdot \omega' = \omega \cdot \omega + 4\omega \cdot x + 4x \cdot x = \omega \cdot \omega + 8x \cdot x \text{ mod } 8$$

so that $\omega \cdot \omega$ is well-defined mod 8.

Proposition 1.13. *Let ω be characteristic for Q. Then $\omega \cdot \omega = \sigma$ mod 8.*

Proof. The form $Q \oplus (1) \oplus (-1)$ is indefinite and odd, and therefore (we shall show in Proposition 1.17) it is isomorphic to $p(1) \oplus q(-1)$ for some p, q where $p - q = \sigma(Q)$. The sum of basis elements is characteristic for $p(1) \oplus q(-1)$, and its square equals $p - q = \sigma(Q)$.

On the other hand, if we take ω characteristic for Q, then $\omega + x + y$ is characteristic for $Q \oplus (1) \oplus (-1)$, where x, y are basis elements for the subspaces corresponding to the forms (1) and (-1). Thus

$$\sigma(Q) = p - q = (\omega + x + y) \cdot (\omega + x + y) = \omega \cdot \omega \text{ mod } 8$$

□

In particular, if Q is even, then $\sigma(Q)$ is divisible by 8.

Example 1.14 (Hyperbolic). The *hyperbolic* form H is even, has rank 2 and signature 0, and is given by the matrix $H := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Example 1.15 (E_8). The E_8 form is the restriction of the Euclidean intersection form to the E_8 lattice in $\mathbb{R}^8$. This is the lattice whose coordinates satisfy the following two conditions:

(1) the coordinates are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$; and
(2) the sum of the coordinates is an even integer.

This is evidently integral. Furthermore, it is unimodular, since $E_8 \cap \mathbb{Z}^8$ has index 2 in both $\mathbb{Z}^8$ and in E_8 itself. Finally, it is even; to see this, choose a basis, e.g. if e_j are the standard basis of $\mathbb{Z}^8$ we can choose the basis

$$(e_{j+1} + e_j) \text{ for } 1 \leq j \leq 7, \ \frac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7 + e_8)$$

The reason for this slightly silly choice is that it gives rise to a matrix

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

which corresponds to the Dynkin diagram of the exceptional Lie algebra E_8.

Remark 1.16. In any $\mathbb{R}^n$ with $2|n$ we can construct a definite lattice E_n generated by the subset of $\mathbb{Z}^n$ with coefficients adding to an even number, and the vector with all coefficients $\frac{1}{2}$. The Euclidean inner product on this lattice is always unimodular. It is integral if $4|n$, and even if $8|n$.

The following proposition gives a complete classification of indefinite integral unimodular forms:

Proposition 1.17. *If Q is indefinite and odd then $Q = p(1) \oplus q(-1)$. If Q is indefinite and even then $Q = aE_8 \oplus bH$ where $b > 0$ and $a = \sigma/8$.*

Proof. Assume first that $\sigma = 0$.

Take x primitive with $x \cdot x = 0$; such an x exists by Theorem 1.12. Since x is primitive, there is y with $x \cdot y = 1$. Let A denote the span of x, y. Then Q restricted to A is unimodular, and therefore the same is true of Q restricted to $A^\perp$, and $Q = Q|A \oplus Q|A^\perp$. The restriction of Q to A is isomorphic to $(1) \oplus (-1)$ if $y \cdot y$ is odd, and to H otherwise. Furthermore, one can check that there is an isomorphism $H \oplus (-1) = 2(-1) \oplus (1)$. Finally, the restriction of Q to $A^\perp$ has signature zero, so the splitting can be continued inductively; this proves the proposition if $\sigma = 0$.

Otherwise, suppose we have two forms Q, Q' both indefinite and of the same rank and type with $\sigma(Q) = \sigma(Q') > 0$. By induction $Q \oplus (-1) = Q' \oplus (-1) = p(1) \oplus q(-1)$ with $p \geq q > 0$. Let x and y be primitive vectors in $p(1) \oplus q(-1)$ spanning the (-1) factors in $Q \oplus (-1)$ and $Q' \oplus (-1)$. They are either both characteristic, or neither is characteristic; in either case one shows directly that there is an automorphism of $p(1) \oplus q(-1)$ taking x to y.

Thus, by induction, if Q is even and indefinite, it is determined by its rank and signature. By Proposition 1.13 we know the signature is divisible by 8, and every possible rank and signature can be achieved by some $aE_8 \oplus bH$. ∎

Example 1.18 (K3). Since the K3 is simply-connected and has $c_1 = 0$, it follows that $w_2 = 0$ and the intersection form is even. Since it is indefinite, of rank 22 and signature -16, it follows that its intersection form is $Q = 2(-E_8) \oplus 3H$.

Example 1.19. Our calculations on the homotopy type of a K3 go through just as easily for a complex hypersurface S_d in $\mathbb{CP}^3$ of any degree d. A suitable Veronese embedding shows that S_d is simply-connected. The Euler characteristic and signature are $(d^2 - 4d + 6)d$ and $(4 - d^2)d/3$ respectively, and Q is even iff d is. Since the intersection form is always indefinite, we can apply Proposition 1.17 to determine it exactly.

Notice if d is even, then σ is divisible by 16 (and not just 8 as follows from the theory of even unimodular forms). This will be explained when we come to prove Rochlin's Theorem 2.1.

1.3.4. *Definite forms.* The classification of definite integral unimodular forms is considerably more complicated. Let Q be definite; by replacing Q by $-Q$ if necessary we assume it is positive definite, so that over $\mathbb{R}$ it is equivalent to the usual inner product on some $\mathbb{R}^n$.

We assume further that Q is even, for simplicity. For an even definite form, σ is equal to the rank n, which is therefore a multiple of 8.

Under the identification of Q with the Euclidean inner product, the domain of Q is identified with some lattice Γ in $\mathbb{R}^n$ of covolume 1. Since Q is unimodular, Γ is isomorphic to its dual.

Associated to Γ there is a *theta function*:

$$\theta_\Gamma(q) := \sum_{x \in \Gamma} q^{(x \cdot x)/2}$$

Notice that the assumptions mean that this is an ordinary Taylor series in q with non-negative integral coefficients that grow like a polynomial of degree $n/2$. If we make the substitution $q = e^{2\pi i \tau}$ then $\theta_\Gamma(i\tau) = \sum_x e^{-\pi\tau(x \cdot x)}$. The *Poisson summation formula* for a function f on $\mathbb{R}^n$ decaying sufficiently fast at infinity, says that for any lattice Λ in $\mathbb{R}^n$, the sum of values of f on Λ equals the sum of values of its Fourier transform on the dual lattice Λ^*.

Since the function $e^{-\pi(x \cdot x)}$ on $\mathbb{R}^n$ is equal to its own Fourier transform, and since Γ is isomorphic to its own dual, the Poisson summation formula implies

$$\theta_\Gamma(-1/\tau) = (i\tau)^{n/2}\theta_\Gamma(\tau)$$

Since also $\theta_\Gamma(\tau + 1) = \theta_\Gamma(\tau)$ and n is a multiple of 8, it follows that $\theta_\Gamma(\tau)$ is a *modular form* of weight $n/2$.

Since as a q-series the constant coefficient of θ_Γ is always equal to 1, the difference of the θ series of any two lattices in the same space is a *cusp form* (i.e. a modular form vanishing at infinity). Since a cusp form of weight k has coefficients which grow like a $(k/2)$th power, it follows that the number $R_\Gamma(m)$ of lattice points x of Γ with $x \cdot x = 2m$ satisfies

$$R_\Gamma(m) = -\frac{2k}{B_k}\sigma_{k-1}(m) + O(m^{k/2})$$

where $k = n/2$ where B_k denotes the kth Bernoulli number, and $\sigma_k(m)$ denotes the sum of the kth powers of the divisors of m.

It is a famous theorem of Siegel (see e.g. [2]) that for each $n = 2k$ divisible by 8, there are only finitely many isomorphism classes of even definite integral unimodular lattices Γ_i in $\mathbb{R}^n$, and if w_i is the order of the automorphism group of Γ_i, there is a formula, the *Siegel mass formula*

$$\sum \frac{1}{w_i}\theta_{\Gamma_i} = m_k E_k$$

where E_k is the Eisenstein series of weight k (normalized to have value 1 at infinity), and m_k is an explicit rational number that can be expressed in terms of a product of Bernoulli numbers.

Since each w_i is at least 2, and the numbers m_k grow very quickly, one knows there are many non-isomorphic forms in high dimensions.

Example 1.20 (dimension 8). The E_8 lattice is the unique definite even unimodular integral form in dimension 8.

Example 1.21 (dimension 16). In dimension 16 there are two isomorphism classes of even lattices, namely $E_8 \oplus E_8$ and E_{16}. Although they are not isomorphic, they have the same theta series (and therefore the same number of vectors of any given norm), since there are no nonzero cusp forms of weight < 12. This implies that the quotient tori $\mathbb{R}^{16}/(E_8 \oplus E_8)$ and $\mathbb{R}^{16}/E_{16}$ are isospectral but not isometric, a famous observation of Milnor [8].

Example 1.22 (dimension 24). There are exactly 24 isomorphism classes of even lattices in dimension 24, classified by Niemeier [10]. Only one of these, the Leech lattice Λ_{24}, has no vector of norm squared equal to 2.

There are at least 80 million isomorphism classes in dimension 32, and at least 10^{51} in dimension 40.

The theta function of an odd definite integral unimodular form is a modular form of 'weight $n/2$' for a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of index 3. There is a similar but more complicated finiteness result in each dimension, and there are many non-isomorphic forms.

1.4. **Homotopy types of 4-manifolds.**

1.4.1. *Handlebodies.* We assume some familiarity with the theory of handlebodies (equivalently, with Morse theory). If M is a smooth n-manifold with boundary, and $S \subset \partial M$ is a smooth $(k-1)$-sphere together with a *framing* (i.e. a trivialization of its normal bundle) we can attach a k-handle to M as follows. The k handle is $D^k \times D^{n-k}$ with boundary $S^{k-1} \times D^{n-k} \sqcup D^k \times S^{n-k-1}$. The $S^{k-1} \times 0$ is the *core* and $0 \times S^{n-k-1}$ is the *co-core*. We attach the k-handle by identifying $S^{k-1} \times 0$ with a tubular neighborhood of S, via the given trivialization of its normal bundle. This produces a new manifold M' (which can be given a smooth structure by 'rounding the corners'), and $\partial M'$ is obtained from ∂M by cutting out a tubular neighborhood of S and gluing in the remaining $D^k \times S^{n-k-1}$; we also call this *surgery* on ∂M.

If M is a smooth manifold, and f is a smooth function with nondegenerate critical values (i.e. the Hessian Hf of f is nondegenerate where $df = 0$), then M has a decomposition into handles, with one i-handle for each critical point of f of index i (i.e. for which Hf has i positive eigenvalues).

1.4.2. *Fundamental groups.*

Theorem 1.23. *Every finitely presented group occurs as the fundamental group of a smooth closed orientable 4-manifold.*

Proof. Let $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$ be a finitely presented group. We start with $S^4 = \partial D^5$ which has trivial π_1. Attaching n 1-handles to D^5 produces a new manifold M with $\partial M = \#^n S^1 \times S^3$ whose fundamental group is free on n generators, which we may identify with the x_i. Choose m disjoint smooth embedded circles α_j in $\#^n S^1 \times S^3$ representing the conjugacy classes r_j. Since M is oriented, the normal bundles to α_j are trivial, so we can attach 2-handles to ∂M to obtain M' by gluing the cores to the α_j. By Seifert-van Kampen, removing a tubular neighborhood of each α_j does not change π_1, but gluing back an $D^2 \times S^2$ kills the element of $\pi_1(\partial M)$ represented by α_j. Thus $\pi_1(\partial M') = G$ as desired. □

Let's assume that G is perfect, so that our manifold has $H_1 = H_3 = 0$. After connect-summing with $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ we obtain a new closed 4-manifold W whose fundamental group is still equal to G, and whose intersection form is indefinite and odd; thus our manifold is homotopy equivalent to some 4-manifold of the form $a\mathbb{CP}^2 \# b\overline{CP}^2$ if and only if G is a presentation of the trivial group. Now, in this case, a theorem of Wall implies that there is an integer N, computable in terms of the presentation of G, so that after taking connect sum with $\#^N S^2 \times S^2$ our manifold becomes *diffeomorphic* to $a\mathbb{CP}^2 \# b\overline{CP}^2 \#^N S^2 \times S^2$. In other words, we have an effective procedure which takes as input a finite presentation of a perfect group, and outputs a pair of smooth, closed, oriented 4-manifolds which are diffeomorphic if and only if the group is trivial. This construction is due to Andrei Markov.

Since by Novikov–Boone the isomorphism problem for finitely presented groups is algorithmically unsolvable, it follows that the diffeomorphism problem for smooth 4-manifolds is also unsolvable. See e.g. Haken [5] for more of the theory of decision problems in topology.

For this reason, one typically studies 4-manifolds under some extra assumption on their fundamental groups, which otherwise are unmanagable. For instance, we will focus largely on the case of *simply connected* 4-manifolds.

1.4.3. *Simply connected 4-manifolds.* The homotopy classification of simply connected 4-manifolds is due to Milnor.

Let W be closed, connected and simply-connected. Then W is oriented, and by Poincaré duality, $H_1 = H_3 = 0$, $H_0 = H_4 = \mathbb{Z}$, and $H_2 = H^2$ is torsion-free of finite rank n. If we let W' be obtained from W by removing a small open ball, then $\partial W' = S^3$, and therefore $H_4(W') = 0$ but otherwise it has the same homology groups as W, and the same (trivial) fundamental group, by Seifert-van Kampen.

Thus $\pi_2(W') = \mathbb{Z}^n$, and there is a map from $\vee_n S^2 \to W'$ inducing isomorphisms in homology in every dimension. By Hurewicz and Whitehead, this map is a homotopy equivalence. Thus W is homotopic to a space $\vee_n S^2 \cup_f D^4$ where $f : \partial D^4 = S^3 \to S^2$ represents some element of $\pi_3(\vee_n S^2)$.

Now, π_3 of a wedge of 2-spheres is free abelian on two kinds of generators:

(1) Elements of $\pi_3(S^2) = \mathbb{Z}$. Here the generator is the Hopf fibration $S^3 \to S^2$.

(2) Whitehead products. Here we think of the copy of $\mathbb{Z}$ generated by the map $S^3 \to S^2 \vee S^2$ obtained as the attaching map of the boundary of the 4-cell for the standard CW structure on $S^2 \times S^2$.

Thus $\pi_3(\vee_n S^2) = \mathbb{Z}^{n(n+1)/2}$ and we can encode the class of f as the entries of a symmetric $n \times n$ matrix Q.

Lemma 1.24. *The matrix Q as above is the intersection form on $H^2(W;\mathbb{Z})$ in the given basis.*

Proof. For each i let $\gamma_i \subset S^3$ be the preimage of a regular value p_i of $f : S^3 \to \vee_n S^2 \to S^2_i$. Then there is an oriented surface $\Sigma_i \subset S^3$ with boundary γ_i, and the projection from the interior of $\Sigma_i \to S^2_i - p_i$ is proper, and degree one near the boundary, and therefore has degree one. Thus we can build a surface A_i in W by pushing Σ_i slightly into D^4, and gluing its boundary to the disk obtained by coning γ_i to the center of D^4.

In this way, the linking number of γ_i and γ_j becomes the (signed) intersection of A_i and A_j when $i \neq j$. For $i = j$ let γ_i and γ'_i be two regular preimages of $p_i, p'_i \in S^2_i$ and observe that the linking number of γ_i and γ'_i is the intersection of the class of A_i with itself. □

In particular, the homotopy type of a simply-connected 4-manifold W can be completely recovered from the intersection form on H^2.

1.4.4. *Freedman's Theorem.* Once we know that simply-connected 4-manifolds are classified up to homotopy by their intersection forms, there are two natural questions:

(1) which forms are realized? and
(2) for a given form that is realized, how many homeomorphism types of simply-connected 4-manifolds realize it? (and more subtly, when is a homotopy equivalence between simply-connected 4-manifolds homotopic to a homeomorphism?)

Freedman's Theorem gives a complete answer to both questions. But to state the theorem we must first say a little about the theory of topological and PL manifolds in various dimensions; see e.g. Rudyak [11] for the most significant details.

Any smooth manifold can be smoothly triangulated, and therefore a smooth manifold has a unique (compatible) PL structure. And one can always forget the smooth or PL structure; thus there are forgetful maps $DIFF \to PL \to TOP$.

The key difference between the PL and the smooth category is that the Alexander trick works in every dimension in the PL world, but not the smooth world; i.e. a PL homeomorphism of D^n which is the identity on the boundary is canonically PL isomorphic to the identity, whereas $\text{Diff}(D^n, \partial D^n)$ is not contractible for large n. In fact, this group is contractible iff $O(n+1) \to \text{Diff}(S^n)$ is a homotopy equivalence, which is proved for $n = 2$ by Smale, and for $n = 3$ by Hatcher (the 'Smale Conjecture'). One can use this to show that every PL manifold of dimension ≤ 5 can be smoothed, and in every dimension ≤ 4 this smoothing is unique. Thus, a topological 4-manifold is (uniquely) smoothable iff it has a PL structure. (these results are not optimal: PL manifolds of dimension ≤ 7 are smoothable, and these smoothings are unique in dimensions ≤ 6).

The tangent bundle of a smooth manifold M is a (real) vector bundle TM, and real vector bundles over any space homotopic to a finite CW complex X are classified up to stable isomorphism by homotopy classes of maps from X to BO, the 'infinite Grassmannian'.

An n-manifold which has merely a topological structure does not *a priori* have a tangent bundle; however it turns out that one can define TM as a *topological* $\mathbb{R}^n$ *bundle*, i.e. a fiber bundle E in the usual sense over M with fiber homeomorphic to $\mathbb{R}^n$, together with a section $M \to E$ (the *zero section*). This can be constructed by taking a suitable open neighborhood of the diagonal in $M \times M$.

If M is PL, then E is a *PL* $\mathbb{R}^n$ *bundle*; i.e. E and M are PL manifolds, and the projection and section are PL maps.

It turns out there are classifying spaces $BTOP$ and BPL for stable isomorphism classes of topological and PL $\mathbb{R}^n$ bundles respectively. There are forgetful maps $BO \to BPL \to BTOP$ which can be thought of as homotopy fibrations. By abuse of notation, one denotes the homotopy fiber of $BPL \to BTOP$ by TOP/PL.

If a topological manifold M admits a PL structure, the classifying map $M \to BTOP$ lifts to BPL. Remarkably, Kirby and Siebenmann showed that if the dimension of M is at least 5, the converse is true. Furthermore, they computed the homotopy type of TOP/PL and showed that it was homotopic to a $K(\mathbb{Z}/2\mathbb{Z}, 3)$. In particular, there is a class $\varkappa \in H^4(BTOP; \mathbb{Z}/2\mathbb{Z})$ such that a topological manifold M of dimension at least 5 admits a PL structure if and only if $f^*\varkappa = 0$, where $f : M \to BTOP$ pulls back the (topological) tangent bundle of M. There are also spaces $BSTOP$ and $BSPL$ that classify oriented bundles, and their homotopy fiber is also TOP/PL.

The class $k := f^*\varkappa$ is called the *Kirby-Siebenmann invariant* of M. If M is a manifold of dimension $n < 5$ one defines k to be the class of $M \times \mathbb{R}^{5-n}$. This is always zero if $n < 4$, since manifolds of dimension ≤ 3 admit unique PL (indeed smooth) structures. Thus for a 4-manifold W, the class k is precisely the obstruction to finding a PL structure on $W \times \mathbb{R}$. By abuse of notation, we denote the number $k[W] \in \mathbb{Z}/2\mathbb{Z}$ also by k. It turns out that k is also precisely the obstruction to finding a PL structure on $W \#^n S^2 \times S^2$ for sufficiently large n.

Now, k is readily seen to be a cobordism invariant. Rochlin's Theorem 2.1, to be proved in the sequel, says that a smooth (and therefore PL) simply-connected 4-manifold with even form has signature a multiple of 16, and since connect summing with $S^2 \times S^2$ does not affect evenness of the form or the signature, it follows that $\sigma(Q)/8$ mod 2 is equal to k.

Theorem 1.25 (Freedman)**.** *Let Q be a unimodular integral form on some $\mathbb{Z}^n$, and let $k \in \mathbb{Z}/2\mathbb{Z}$, where if Q is even, we have $k = \sigma(Q)/8$ mod 2. Then there exists a closed, oriented simply-connected 4-manifold W with intersection form Q and Kirby-Siebenmann invariant k.*

Furthermore, if W, V are closed, simply-connected 4-manifolds, if $h : H_2(W; \mathbb{Z}) \to H_2(V; \mathbb{Z})$ is an isomorphism preserving intersection forms, and if $k(W) = k(V)$, then there is a homeomorphism $f : W \to V$, unique up to isotopy, so that $f_* = h$ on H_2.

2. Rochlin's Theorem

The goal of this section is to prove the following theorem of Rochlin:

Theorem 2.1 (Rochlin)**.** *Let W be a smooth, simply-connected 4-manifold with even intersection form. Then the signature σ is divisible by 16.*

Freedman's Theorem 1.25 provides many examples of topological simply-connected 4-manifolds with even intersection form and σ equal to 8 mod 16. Thus Rochlin's theorem necessarily uses the smooth structure in an essential way.

Rochlin's original proof uses deep results from algebraic topology. Instead, we give a more direct proof using Borel and Hirzebruch's $\hat{A}$ genus and the Atiyah–Singer Index Theorem.

Here is the sketch of the proof. We will recall the definition of the *Dirac operator*, a certain first order linear operator that was introduced to give a relativistic quantum theory of the electron. This operator makes sense on a spin manifold of any dimension, and depends on the choice of an auxiliary Riemannian metric. The Dirac operator is *elliptic*, which implies that its kernel and cokernel are finite dimensional (complex) vector spaces. The dimensions of these spaces depend on the choice of metric, but their difference — the *index* — does not, and is a smooth invariant called the $\hat{A}$ *genus*.

Since it counts dimensions, the $\hat{A}$ genus is an integer. However, in dimensions congruent to 4 mod 8, the kernel and cokernel of the Dirac operator have the structure of vector spaces over the *quaternions*, and therefore their dimension as complex vector spaces is *even*.

Now, in dimension 4, the $\hat{A}$ genus and the signature are proportional, with constant of proportionality $-1/8$. Since the $\hat{A}$ genus is an even integer, this implies the signature is divisible by 16.

2.1. **Hodge theory.** Inspired by Riemann's use of harmonic functions in his theory of abelian integrals, Hodge began around 1930 the systematic study of harmonic forms on Riemannian manifolds.

If M is a smooth, closed n-manifold, a Riemannian metric g on the tangent bundle TM defines pointwise a metric on every natural tensor bundle on M, in particular on each $\Omega^p(M)$. If M is oriented, the volume form $\text{vol} \in \Omega^n(M)$ is is a nowhere zero section, and therefore gives a trivialization of, the line bundle $\Lambda^n M$ of top dimensional forms.

In terms of coordinates, if e_i is an orthonormal basis for T_xM, then the set of $e_{i_1}\wedge\cdots\wedge e_{i_p}$ over all $i_1 < \cdots < i_p$ is an orthonormal basis for $\Lambda^p T_x M$.

This metric lets us define a symmetric pairing on $\Omega^p(M)$, given by

$$\langle \alpha, \beta\rangle := \int_M g(\alpha,\beta)\text{vol}$$

2.1.1. *Hodge star.* If $\alpha \in \Omega^p(M)$ is any form, Hodge defines a *dual* form $*\alpha$ defined pointwise by

$$\beta \wedge *\alpha := g(\alpha,\beta)\text{vol}$$

for all p-forms β. Note that $*^2 = (-1)^{p(n-p)}$ on p-forms.

Then for a p-form α and a $(p-1)$-form β we have

$$\langle \alpha, d\beta\rangle = \int \alpha\wedge *d\beta = \int (-1)^{p^2} d\beta \wedge *\alpha = \int (-1)^{p^2+1}\beta\wedge d*\alpha = \langle \beta, -*d*\alpha\rangle$$

so that $\delta := -*d*$ is a formal adjoint to d.

2.1.2. *The Laplacian.* The (Hodge) Laplacian is the operator $\Delta := d\delta + \delta d$. For each p, it acts as a non-negative symmetric self-adjoint operator $\Delta : \Omega^p(M) \to \Omega^p(M)$. A p-form α is *harmonic* if $\Delta\alpha = 0$.

2.1.3. *Elliptic PDE.* The equation $\Delta\alpha = 0$ is a second-order linear PDE. Its key analytic property is that it is *elliptic*. To explain what this means, we recall the definition of the *symbol* of a differential operator. If E, F are vector bundles over M, and $P : \Gamma(E) \to \Gamma(F)$ is a linear operator of order k, then if x denote local coordinates on M, and $u \in \Gamma(E)$, we can write locally

$$Pu = \sum_{|I|=k} P^I(x)\partial_I u + \text{ lower order terms}$$

where $P^I : E_x \to F_x$ is linear, and symmetric in the indices of I.

The chain rule implies that the highest order coefficients — i.e. P^I with $|I| = k$ — transform like a symmetric tensor of order k, and we therefore obtain a map, called the *symbol*:

$$\sigma(P) : S^k(T^*M) \otimes E \to F$$

With this definition, we say P is *elliptic* if $\sigma(P)$ is invertible; i.e. for each nonzero $\theta \in T_x^*M$ the map $\sigma(P)(\theta, \cdots, \theta) : E_x \to F_x$ is invertible.

If $E = F$ and P is second order, this is equivalent to the condition that $\sigma(P)$ defines a positive-definite inner product on T_x^*M for each x. Now, the definition of the Laplacian Δ uses the Riemannian metric, but is otherwise not coordinate dependent; it follows that the symbol of Δ is invariant under the orthogonal symmetries of $S^2(T_x^*M)$. But the Riemannian metric itself is the only invariant vector in $S^2(T_x^*M)$ up to scale, and is therefore proportional to $\sigma(\Delta)$. In particular, we conclude that Δ is elliptic.

Using the Fourier transform and the Sobolev embedding theorem one shows that an elliptic operator is *Fredholm*; i.e. it has finite dimensional kernel and cokernel. Thus in particular the space of harmonic p-forms on a Riemannian manifold is finite dimensional.

A more precise analysis shows that the spectrum of Δ in each dimension is discrete and non-negative, that the eigenspaces are finite dimensional, and that $\Omega^k(M)$ has an orthogonal decomposition into these eigenspaces.

Example 2.2. A *wave packet* is a function ψ which is localized (to the extent that this is possible) in both space and momentum; e.g. $\psi := \eta(x/\epsilon)e^{i(\xi\cdot x)/\hbar}$ where η is a smooth cutoff function, where ξ is the momentum (one can think of ξ as a cotangent vector at 0) and where $0 < \hbar \ll \epsilon \ll 1$.

A wave packet is, to first order, an eigenfunction for a differential operator, and the symbol 'is the eigenvalue'. The extent to which position and momentum can be simultaneously localized is, of course, controlled by the size of $\hbar$.

2.1.4. *The Hodge Theorem.* Once one knows that the space $\mathcal{H}^p(M)$ of harmonic p-forms on a Riemannian manifold M is finite dimensional, it is natural to wonder to what extent this dimension depends on the metric. Hodge's fundamental discovery is that the dimension of $\mathcal{H}^p(M)$ is *independent* of the metric, and in fact the space of harmonic forms can be (canonically) identified with the (real) de Rham cohomology of M (itself canonically isomorphic to the real singular cohomology of M considered purely as a topological space). In more detail, one has

Theorem 2.3 (Hodge Theorem)**.** *Let M be a smooth, closed, oriented n-manifold. Then for each p there is an orthogonal decomposition*

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathcal{H}^p(M)$$

Furthermore, the harmonic forms are closed, and the inclusion $\mathcal{H}^p(M) \to \Omega^p(M)$ induces an isomorphism $\mathcal{H}^p(M) \to H^p_{dR}(M;\mathbb{R})$.

Let us remark that the difficulty in this theorem is purely analytic, and is solved by the theory of elliptic regularity, as outlined at the end of § 2.1.3. After this, the proof is purely formal, and proceeds as though the spaces $\Omega^p(M)$ were finite dimensional. We outline this formal argument now.

Suppose we have a chain complex of *finite dimensional* Hilbert spaces

$$\cdots E^{k-1} \xrightarrow{d} E^k \xrightarrow{d} E^{k+1} \cdots$$

and let $\delta : E^k \to E^{k-1}$ and so on denote the adjoint operators. Finally, let $\Delta := d\delta + \delta d$ and denote $\ker\Delta$ by $\mathcal{H}^k$ in each dimension.

In this context the analog of the Hodge theorem is elementary: there is an orthogonal decomposition in each dimension

$$E^k = dE^{k-1} \oplus \delta E^{k+1} \oplus \mathcal{H}^k$$

and the inclusion $\mathcal{H}^k \to E^k$ induces an isomorphism $\mathcal{H}^k \cong \ker d|_{E^k}/dE^{k-1}$.

This follows from a sequence of elementary observations:

(1) *A form α is in* $\ker\Delta$ *if and only if it is in* $\ker d$ *and in* $\ker\delta$*.* This follows from the adjoint property:

$$\langle\Delta\alpha,\alpha\rangle = \langle d\alpha, d\alpha\rangle + \langle\delta\alpha,\delta\alpha\rangle = \|d\alpha\|^2 + \|\delta\alpha\|^2$$

(2) *For each k we have $E^k = \ker\Delta \oplus \operatorname{im}\Delta$.* For, Δ is self-adjoint, and therefore E^k has an orthogonal decomposition into eigenspaces. But then $\ker\Delta$ is the 0 eigenspace, and $\operatorname{im}\Delta$ is the sum of the nonzero eigenspaces.

(3) *For each k, the three spaces* $\ker\Delta$*, dE^{k-1}, δE^{k+1} are orthogonal.* For, if $\Delta\alpha = 0$ then $d\alpha = 0$ so

$$\langle\alpha,\delta\beta\rangle = \langle d\alpha,\beta\rangle = 0$$

and similarly $\langle\alpha, d\beta\rangle = 0$. Likewise,

$$\langle d\beta,\delta\gamma\rangle = \langle d^2\beta,\gamma\rangle = 0$$

(4) *For each k we have* $\operatorname{im}\Delta = dE^{k-1} \oplus \delta E^{k+1}$. Evidently $\operatorname{im}\Delta$ is contained in the RHS. But conversely the RHS is perpendicular to $\ker\Delta$, and therefore contained in $\operatorname{im}\Delta$.

2.1.5. *(Anti-)Self Duality.* Since $*^2 = (-1)^{p(n-p)}$ on p-forms, it follows that when n is of the form $4k$ and $p = 2k$ that $*^2 = 1$, and there is a decomposition of the middle dimensional forms into ±1 eigenspaces of $*$. If α is harmonic, then so is $*\alpha$, since to be harmonic is equivalent to being in the kernel of both d and $d*$. Thus $*$ acts as an involution on $\mathcal{H}^{2k}$, and there is a decomposition into eigenspaces $\mathcal{H}^{2k} = \mathcal{H}^{2k}_+ \oplus \mathcal{H}^{2k}_-$. Identifying $\mathcal{H}^{2k}$ with H^{2k}_{dR} identifies wedge product of forms with the intersection pairing Q on cohomology classes.

Thus the ± 1 eigenspaces of $*$ correspond to the orthogonal decomposition into positive definite and negative definite subspaces of Q. In other words,

$$\dim \mathcal{H}^{2k}_+ - \dim \mathcal{H}^{2k}_- = b^{2k}_+ - b^{2k}_- = \sigma$$

On a 4-manifold we have $\dim \mathcal{H}^2_+ - \dim \mathcal{H}^2_- = \sigma$. Every 2-form can be written as a linear combination of *self dual* and *anti-self dual* 2-forms. If $e_1, \cdots, e_4$ is an oriented basis for the cotangent space at some point, a basis[1] for the self-dual 2-forms Λ^2_+ is given by

$$e_1 \wedge e_2 + e_3 \wedge e_4, \quad e_1 \wedge e_3 + e_4 \wedge e_2, \quad e_1 \wedge e_4 + e_2 \wedge e_3$$

and for the anti-self-dual 2-forms Λ^2_- by

$$e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3 - e_4 \wedge e_2, \quad e_1 \wedge e_4 - e_2 \wedge e_3$$

This decomposition illustrates a special feature of 4 dimensional geometry. The Riemannian metric gives rise to an action of $\mathrm{SO}(4,\mathbb{R})$ on Λ^2 pointwise, which commutes with $*$. Thus there is a homomorphism from $\mathrm{SO}(4,\mathbb{R})$ to $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$, acting by isometries on the individual $\Lambda^2_\pm$ factors. This homomorphism is surjective, and since both Lie groups are 6 dimensional it follows that it is a covering map. Since $\pi_1(\mathrm{SO}(4,\mathbb{R})) = \pi_1(\mathrm{SO}(3,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ it follows that this is a 2-fold cover, with kernel $\pm\mathrm{id}$. That is, there is a short exact sequence

$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{SO}(4,\mathbb{R}) \to \mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$$

i.e. the Lie algebra $\mathfrak{so}(4,\mathbb{R})$ is *not* simple, but is a sum of two copies of $\mathfrak{so}(3,\mathbb{R})$.

2.2. **Quantum mechanics.**

2.2.1. *Classical mechanics.* In the Hamiltonian formulation of classical mechanics, the instantaneous state of a system is described by a point in a symplectic manifold (usually T^*M for some smooth M with its canonical symplectic form ω), the total energy (i.e. the sum of kinetic and potential energy) is a function H on T^*M called the *Hamiltonian*, and the system evolves by the flow of the vector field X_H defined by $\omega(X_H, \cdot) := dH$. Notice that $X_H(H) = 0$; i.e. energy is *conserved*.

The same formula $\omega(X_f, \cdot) := df$ associates a vector field X_f to any smooth function f. Lie bracket of vector fields induces in this way a Lie bracket on $C^\infty(T^*M)$ called the *Poisson bracket*, and defined by

$$\{f, g\} := \omega(X_f, X_g)$$

For any f, the evolution of f under the Hamiltonian flow satisfies

$$\frac{df}{dt} = X_H(f) = \{f, H\}$$

Position and momentum together define *conjugate* coordinates on T^*M whose respective level sets give complementary Lagrangian foliations. Locally we can choose position coordinates x_i and momentum coordinates p_i so that the symplectic form is $\omega := \sum dx_i \wedge dp_i$ and $\{x_i, p_j\} = \delta_{ij}$.

[1] note these basis elements have length $\sqrt{2}$

2.2.2. *Quantization.* In (first) quantization, the (smooth) functions on our symplectic manifold are replaced by *Hermitian operators* on a *Hilbert space* V, and the Poisson bracket on functions is replaced by the *commutator* of operators. In the case of a cotangent bundle T^*M the Hilbert space is $L^2(M)$.

One introduces a formal variable $\hbar$ called *Planck's constant* (assumed in practice to be very small) so that if functions $f, g, \{f, g\}$ on the symplectic manifold are replaced by operators $Q_f, Q_g, Q_{\{f,g\}}$, then we should have

$$[Q_f, Q_g] := Q_f Q_g - Q_g Q_f = i\hbar Q_{\{f,g\}} + O(\hbar^2)$$

If $\psi \in V$ denotes the instantaneous state of the system, and H is the Hamiltonian (i.e. the energy of the system), the evolution in time is given by the equation

$$i\hbar \frac{d\psi}{dt} = Q_H \psi$$

The (complex-valued) function ψ is called the *wave function* of the system. We interpret the non-negative function $\|\psi\|^2$ as the *probability* of finding the system in a particular location (in M). Thus we should insist that $\int_M \|\psi\|^2 d\text{vol} = 1$. Since Q_H is Hermitian, $-iQ_H/\hbar$ is skew-Hermitian, so that the evolution of ψ is *unitary*, and the condition that the total probability is 1 is preserved.

The quantization $f \to Q_f$ is not unique, but sometimes there is a natural choice. In the case that the classical phase space is a cotangent bundle T^*M and the Hilbert space is $L^2(M)$, the *standard representation* is defined as follows:

(1) for f a function on T^*M of the position x alone, we define

$$Q_f \psi = f\psi$$

and

(2) for $f = p_i$, one of the momentum coordinates (corresponding to dx_i) we define

$$Q_{p_i} \psi = -i\hbar \frac{\partial \psi}{\partial x_i}$$

For an operator associated to a more general function of x and p we must expand as a power series in p.

We remark that even in the most 'elementary' cases, e.g. where $M = \mathbb{R}^n$, the operators Q_{x_i} and Q_{p_i} associated to the position and momentum coordinates are *unbounded*.

2.2.3. *Schrödinger equation.* In fact the time evolution equation does not need to be put in by fiat, but emerges naturally from the standard representation.

Noether's theorem identifies the momenta p_i as conserved quantities associated to invariance of the classical equations of motion under translation in (flat) space. Analogously, energy E is a conserved quantity associated to the invariance under translation in *time*. Thus on $\mathbb{R}^4$ with space coordinates x_1, x_2, x_3 and time coordinate t we have the associated momenta p_1, p_2, p_3, E.

Total energy is kinetic plus potential energy. For a single particle of mass m and velocity v, the kinetic energy is $(1/2)mv^2 = \sum_i 1/(2m)p_i^2$. If the particle moves in a potential of

$V(x,t)$, we have the equation

$$E = \sum \frac{1}{2m} p_i^2 + V(x,t)$$

Under the standard representation, E becomes the operator $i\hbar\partial_t$.[2] Thus, if ψ denotes the state of the quantized particle, we obtain the *Schrödinger equation*:[3]

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{-\hbar^2}{2m} \Delta + V(x,t) \right) \psi$$

2.3. **Spinors.**

2.3.1. *Special Relativity and the Klein–Gordon equation.* In special relativity the speed of light c is constant in all inertial reference frames. This means that physics is invariant under affine transformations of 'spacetime' $\mathbb{R}^4$ that preserve the metric $dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2$. The group of linear transformations of this form is called the *Lorentz group*.

In order for momentum to be preserved in any inertial frame, it is necessary to modify (or reinterpret) the formula $p = mv$ as

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

where m_0 is 'rest mass' (i.e. mass as measured in a frame for which the object is stationary), and v is velocity.

Remark 2.4. Note that in units with $c = 1$, this is the usual formula (up to scale) for the Poincaré metric in the unit disk, explaining the coincidence that the Lorentz group is the isometry group of hyperbolic 3-space.

Now, the expression pc has units of energy; in fact, this is the classical formula for 'radiation pressure' of electromagnetic radiation. For $v \ll c$ it approximates to $(1/2)m_0 v^2 + m_0 c^2$. Interpreting $m_0 c^2$ as the 'rest energy', from Einstein's famous formula, it is natural to take pc as an approximation for total energy, at least in the absence of a potential, and when velocity is small. Now, pc is not Lorentz invariant, but by considering the correction term, one arrives at

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

as a Lorentz invariant expression for total energy E.

Under the standard representation, this quantizes to give the *Klein–Gordon equation*:

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + \frac{m_0^2 c^2}{\hbar^2} \psi = 0$$

[2] Note the sign! Disclosure: I'm not entirely sure where this sign comes from, unless it's from the Lorentz signature of spacetime in special relativity giving Newtonian physics as a limit as $c \to \infty$.

[3] Δ in this equation is the *analyst's Laplacian*, which has the opposite sign to the *geometer's Laplacian* defined in § 2.1.

2.3.2. *Dirac Equation.* The Klein–Gordon equation, while relativistically invariant, and superficially similar to the Schrödinger equation, is unsatsifactory as a quantum theory of the electron. The main difficulty is that it is *second order* in the time variable. This means that one can give arbitrary initial values of ψ and $\partial_t \psi$, and therefore even locally, in a regime where $v \ll c$, we cannot use the equation to determine the evolution of an initial state ψ. In particular, there is no reason for the evolution of ψ to be unitary, and we cannot interpret $\|\psi\|^2$ any more as a probability.

Ignoring mass for the moment, and setting $c = 1$, the Klein–Gordon equation can be solved by finding a basis of eigenfunctions ψ for the operator $\Delta - \partial_t^2$. Dirac looked for a factorization of the operator $\Delta - \partial_t^2$ of the form

$$\Delta - \partial_t^2 = (A\partial_x + B\partial_y + C\partial_z + iD\partial_t)^2$$

Thus we must have $A^2 = 1$ and likewise for B, C, D. But also, since cross-terms must cancel, we must have $AB + BA = 0$ and so on. Dirac [3] realized this could be accomplished if A, B, C, D were *matrices*[4], for example

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus ψ is not a function on space-time, but a section of a $\mathbb{C}^4$ bundle.

The first two coordinates transform in the correct way to describe the *spin* of an electron, solving a puzzle for the Schrödinger equation after the discovery of spin in the Stern–Gerlach experiment. The second two coordinates also describe the spin of a particle with the same mass as the electron, but with opposite electric charge; Dirac interpreted this as predicting the existence of a new particle, which has come to be known as the *positron*, and whose existence was confirmed by Carl Anderson in 1932.

However this leads to a new puzzle. Dirac's equation says that the positron should have eigenstates of *negative energy* (these come from the diagonal -1s in the matrix representing D above). This is physically unnatural, since any given state of a positron will be excited, and it should endlessly radiate, and fall to states of more and more negative energy and higher and higher momentum (this is a problem with the Klein–Gordon equation too). Dirac's 'solution' to this puzzle involves an interpretation of a positron as a 'hole' in a 'Dirac sea' of invisible particles, in which all negative energy states are already filled (and are therefore inaccessible by the Pauli exclusion principle). One of these invisible particles can be excited e.g. by a photon, jumping out of its negative energy state, and simultaneously creating a hole in the negative energy sea (a positron) and a positive energy state (an electron). Conversely, an electron-positron pair can annihilate each other, releasing the excitation energy as a photon.

The spontaneous creation or annihilation of a particle cannot be accounted for in ordinary quantum mechanics, where the Hilbert space of the theory encodes the degrees of freedom of a fixed number of particles. The conclusion is that ordinary quantum mechanics is ultimately incompatible with special relativity, and one needs *quantum field theory*.

[4] why 4×4 matrices? we shall see why.

2.4. **Dirac equation on a Riemannian manifold.**

2.4.1. *Clifford algebra.* Let's return to the geometer's Laplacian, which on ordinary Euclidean space has the form $\Delta = -\sum_i \partial_i^2$. We denote the Dirac operator (following Feynman) by $\not{\partial}$, and observe that in these coordinates, $\not{\partial} := \sum_i e_i \cdot \partial_i$ for suitable e_i satisfying $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$, so that $\not{\partial}^2 = \Delta$ (the choice of sign ensures that $\not{\partial}$ has totally real spectrum).

How does this expression transform under a (linear) change of coordinates? The definition of the Laplacian on a smooth manifold requires an orientation and a Riemannian metric; thus we should only allow coordinate transformations by elements of $\mathrm{SO}(n;\mathbb{R})$. The ∂_i transform like vectors (since that's what they are); thus for $\not{\partial}$ to be coordinate independent, the e_i must transform like *covectors*. In other words, we can take $e_i = dx_i$ and think of the new multiplication rule as a *deformation* of the usual exterior product on covectors.

Extending by linearity and associativity, we get a new algebra structure on the space $\oplus_i \Lambda^i T_p^*$ of forms at a point p. This algebra has dimension 2^n, and is called the *Clifford algebra* Cl_n. The classification over the complex numbers is easy, modulo the Artin-Wedderburn theorem: if $n = 2m$ is even, Cl_n is isomorphic to the algebra of $2^m \times 2^m$ complex matrices. If $n = 2m + 1$ is odd, Cl_n is isomorphic to the direct sum of *two copies* of the algebra of $2^m \times 2^m$ complex matrices. Thus in either case we can think of the e_i as (matrix) operators on a complex vector space V of dimension $2^m \times 2^m$.

2.4.2. *Representation theory.* The action of the orthogonal group $\mathrm{SO}(n,\mathbb{R})$ induces automorphisms of Cl_n. How does $\mathrm{SO}(n,\mathbb{R})$ act on V? Perhaps surprisingly, it doesn't! To see this in dimension 4, let's return to the matrices A, B, C, D from § 2.3.2. The vector space V decomposes into 2 dimensional ± 1 eigenspaces of D. Let's focus on the $+1$ eigenspace. This decomposes in turn into $\pm i$ eigenspaces for the operator AB with eigenvectors that we denote (with physics in mind) by $|+\rangle$ and $|-\rangle$:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } |-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Consider the effect of rotation by θ in the x–z plane; this fixes B and replaces A by $A_\theta := \cos(\theta)A + \sin(\theta)C$. The operator $A_\theta B$ still decomposes the $+1$ eigenspace of D into $\pm i$ eigenspaces $|+_\theta\rangle$ and $|-_\theta\rangle$, but for small θ we see that

$$|+_\theta\rangle \text{ is proportional to } \begin{pmatrix} 1 \\ (1-\cos(\theta))/\sin(\theta) \\ 0 \\ 0 \end{pmatrix}$$

which for θ small, is approximately equal to $\cos(\theta/2)\,|+\rangle + \sin(\theta/2)\,|-\rangle$.

In other words: rotating space in the x–z plane through angle θ 'rotates' the $+1$ eigenspace of D through the angle $\theta/2$. This is the famous observation that if an electron is rotated through 360°, the phase of its spin is multiplied by -1.[5]

For the electron, this means the structure group of the $\mathbb{C}^4$ bundle where the wave function takes values is not the Lorentz group $\mathrm{SO}(3,1)$ but its double cover $\mathrm{SL}(2,\mathbb{C})$. In the

[5] this does *not* mean that spin 'up' changes to spin 'down', but rather that the complex *amplitude* of the spin components are multiplied by -1

Euclidean case (or on a Riemannian manifold), V is not a module for $\mathrm{SO}(n,\mathbb{R})$, but for its double cover $\mathrm{Spin}(n)$. Note $\mathrm{Spin}(n)$ is simply-connected if $n > 2$.

Example 2.5. When $n = 3$ we have $\mathrm{SO}(3,\mathbb{R}) = \mathbb{RP}^3$ so that $\mathrm{Spin}(3) = S^3 = \mathrm{SU}(2)$. The action of $\mathrm{Spin}(n)$ on V is the usual action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$.

Example 2.6. When $n = 4$ we have $\mathrm{Spin}(4) = S^3 \times S^3$ and $\mathrm{SO}(4,\mathbb{R})$ is the quotient by the diagonal $\mathbb{Z}/2\mathbb{Z}$. To see this, we think of $\mathbb{R}^4$ as the quaternions, and identify each of the two copies of S^3 in $\mathrm{Spin}(4)$ with the unit quaternions.

Given $(q_1, q_2) \in S^3 \times S^3$ and a quaternion $v \in \mathbb{R}^4$ we define

$$(q_1, q_2) \cdot v = q_2 v q_1^{-1}$$

To see that this surjects onto $\mathrm{SO}(4,\mathbb{R})$, observe that $(q, 1)$ acts by rotating two orthogonal 2-planes in $\mathbb{R}^4$ through the same (signed) angle, whereas $(1, q)$ acts by rotating the same orthogonal 2-planes through the same angle with opposite signs.

Evidently $(-1, -1)$ is in the kernel. Since $S^3\times S^3$ and $\mathrm{SO}(4,\mathbb{R})$ have the same dimension, this action gives a covering map; and since $\pi_1(\mathrm{SO}(4,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ it follows that this map is the universal cover, and $S^3 \times S^3 = \mathrm{Spin}(4)$ as claimed.

2.4.3. *Decomposition of V.* We suppose in this section that $n = 4$. Even though $V = \mathbb{C}^4$ is irreducible as a module for Cl_4, the action of the 1-forms alone preserves a splitting of V as $V^+ \oplus V^-$, where multiplication by a 1-form interchanges the two factors.

In local coordinates, if $e_1, \cdots, e_4$ is an orthonormal basis for T_p^*, we can choose coordinates on V for which Clifford multiplication is represented by the matrices

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

This gives the splitting $V = V^+ \oplus V^-$ into 2×2 blocks, and we see immediately that Clifford multiplication by a 1-form takes $V^\pm \to V^\mp$.

As representations of $\mathrm{Spin}(4) = S^3 \times S^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$ the factors $V^\pm$ correspond to the standard 2-dimensional representations of the $\mathrm{SU}(2)$ factors. This gives us another way to define V intrinsically. Recall that $\Lambda^2 \cong \mathbb{R}^6$ decomposes under $*$ into 3 dimensional ± 1 eigenspaces $\Lambda^2_\pm$, and $\mathrm{SO}(4,\mathbb{R})$ acts on these factors under the 2-fold covering map $\mathrm{SO}(4,\mathbb{R}) \to \mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$. Under the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ we can replace each $\mathbb{R}^3$ with a copy of $\mathbb{C}^2$; these two $\mathbb{C}^2$s are $V^\pm$.

2.4.4. *Spin structure and w_2.* On an oriented Riemannian manifold M, the metric turns each $\oplus_i \Lambda^i T_p^* M \otimes \mathbb{C}$ into a copy of the (complex) Clifford algebra Cl_n. We would like an associated *spinor bundle* S over M with fiber a complex vector space of dimension $\mathbb{C}^m$ corresponding fiberwise to the realization of Cl_n as a matrix algebra. This can always be accomplished locally, but globally there is a topological obstruction.

On a 4-manifold, we can proceed as before: if $e_1, \cdots, e_4$ is a local basis of orthonormal 1-forms, then e_1 (say) has eigenvalues $\pm i$, corresponding to 2-dimensional eigenspaces; the same is true of the operator e_2e_3. But e_1 commutes with e_2e_3, and we can find a basis of simultaneous orthonormal eigenfunctions for these two operators, giving a local trivialization of S. A similar construction works in any dimension.

On the overlap of two charts one basis of 1-forms transforms relative to another pointwise by an element of $\mathrm{SO}(n,\mathbb{R})$. We may choose one of two preimages of this element in $\mathrm{Spin}(n)$ to glue adjacent charts of S on the overlaps, but this gluing must be compatible on triples of charts to give a well-defined bundle globally; thus the obstruction is a 2-cocycle, which is to say an element of $H^2(M;\mathbb{Z}/2\mathbb{Z})$. A (stable) trivialization of TM (or T^*M) over the 2-skeleton lets us solve this gluing problem, so we can identify this obstruction with w_2, the second Stiefel–Whitney class. In short: the bundle S of spinors exists providing M is oriented, and $w_2 = 0$. If M is 4-dimensional and simply-connected, this is equivalent to having an even intersection form.

2.4.5. *The Dirac operator on a Riemannian manifold.* Thus if M is an oriented Riemannian manifold, the differential forms $\Omega^*(M)$ can be given the structure of a bundle of Clifford algebras, and if $w_2(M) = 0$ there is a $\mathbb{C}^m$ bundle S (whose sections ψ are *spinors*) on which this Clifford algebra acts pointwise by the standard representation.

To define the Dirac operator we need to be able to make sense of the derivative of a spinor ψ along a tangent vector to M; i.e. we need a *connection* on S. The Riemannian metric determines a canonical connection on TM, the *Levi–Civita connection*, and thereby a canonical connection on T^*M.

We define a connection on S as follows. If γ is a smooth curve in M, we can pick an orthonormal basis e_i for T^*M at some point on γ and parallel transport it along the curve to get a family of bases. As above we get a family of orthonormal bases of eigenvectors for W along γ. Declaring this family to be parallel defines implicitly a connection, and therefore we can make sense of covariant derivatives $\nabla_i\psi$. Now define

$$\partial\!\!\!/\psi := \sum_i e_i \cdot \nabla_i \psi$$

The symbol of $\partial\!\!\!/$ is $\sigma(\partial\!\!\!/)(\theta, w) = \theta \cdot w$ where $w \in S_p$, $\theta \in T_p^*M$ and $\cdot$ denotes Clifford multiplication. Since for any 1-form θ Clifford multiplication is an automorphism where θ is nonzero, it follows that $\partial\!\!\!/$ is elliptic. Furthermore, the symbol of $\partial\!\!\!/^2$ is $\sigma(\partial\!\!\!/^2)(\theta, w) = -\|\theta\|^2 w$ (thus explaining why $\partial\!\!\!/$ is a kind of 'square root' of the Laplacian).

On a smooth spin 4-manifold W, as in § 2.4.3 we can decompose the $\mathbb{C}^4$ bundle S naturally into a direct sum of two $\mathbb{C}^2$ bundles $S = S^+ \oplus S^-$ which are interchanged by Clifford multiplication by a 1-form. Since this decomposition is natural with respect to the orthogonal structure, it is preserved by the covariant derivative. Thus, if $\psi \in \Gamma(S^+)$, we have $\partial\!\!\!/\psi \in \Gamma(S^-)$, and conversely.

These bundles can be constructed more functorially from the bundle of 2-forms $\Lambda^2(W)$ with its splitting into self dual and anti-self dual 2-forms $\Lambda^2_\pm(W)$. These bundles have structure group $\mathrm{SO}(3,\mathbb{R})$, corresponding to the two factors of $\mathrm{SO}(4,\mathbb{R})$; a spin structure on W lets us lift the structure group of these bundles to $\mathrm{SU}(2)$, and construct the associated $\mathbb{C}^2$ bundles $S^\pm$.

The Dirac operator is self-adjoint; if we denote by $\partial\!\!\!/^\pm$ the restrictions of $\partial\!\!\!/$ to $\Gamma(S^\pm)$ then these operators are the adjoints of each other. Since $\partial\!\!\!/$ is elliptic, so are $\partial\!\!\!/^\pm$, and they have finite dimensional kernels and cokernels.

By the adjoint property, the cokernel of $\partial\!\!\!/^+$ is equal to the kernel of $\partial\!\!\!/^-$, and we can therefore compute the *index* of $\partial\!\!\!/^+$ by

$$\text{index of } \partial\!\!\!/^+ = \dim\ker \partial\!\!\!/^+ - \dim\ker \partial\!\!\!/^-$$

2.5. **Atiyah–Singer Index Theorem.** Let E, F be smooth complex bundles over an oriented smooth manifold M, and let $D : \Gamma(E) \to \Gamma(F)$ be a linear elliptic partial differential operator. The kernel and cokernel of D are finite dimensional, and the *analytic* index of D is defined to be the difference $\text{index}(D) := \dim\ker D - \dim \text{coker} D$.

The Atiyah–Singer Index Theorem gives an equality between this analytic index and a *topological* index, defined purely in terms of characteristic class data associated to M, E, F, D.

More generally, we can consider a *complex* of vector bundles E^j and linear operators d_j (of the same order):

$$0 \to \Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Gamma(E^n) \to 0$$

We can pull back each bundle E^j to T^*M under the projection π, and obtain a complex of vector bundles called the *symbol complex*

$$0 \to \pi^*E^0 \xrightarrow{\sigma(d_0)} \pi^*E^1 \xrightarrow{\sigma(d_1)} \cdots \xrightarrow{\sigma(d_{n-1})} \pi^*E^n \to 0$$

We say that the original complex is *elliptic* if the associated symbol complex is *exact* away from the zero section of T^*M.

We can define $E^{\text{even}} = \oplus E^{2j}$ and $E^{\text{odd}} = \oplus E^{2j+1}$ and define $D := \sum d_{2j} + d^*_{2j+1}$. Then

$$0 \to \Gamma(E^{\text{even}}) \xrightarrow{D} \Gamma(E^{\text{odd}}) \to 0$$

is elliptic in the usual sense, and the index of D is the *Euler characteristic* of the (cohomology of the) original complex $\Gamma(E^*)$.

2.5.1. *K-theory.* Recall that on a compact Hausdorff topological space X the set of isomorphism classes of finite dimensional complex vector bundles on X forms an abelian semiring with $\oplus$ as sum, and $\otimes$ as multiplication. Following Grothendieck, one obtains a ring $K(X)$ whose elements are ordered pairs of isomorphism classes of bundles (E, F) modulo the equivalence relation $(E_0, F_0) \sim (E_1, F_1)$ if there is G so that $E_0 \oplus F_1 \oplus G = E_1 \oplus F_0 \oplus G$. It is usual to write $(E, 0) = [E]$ and $(0, E) = -[E]$ and $(E, F) = [E] - [F]$.

On a *noncompact* space X one has to work harder to define K-theory in general (e.g. because of phenomena such as swindles) but it is straightforward to define K theory with *compact support*. We denote this $K_c(X)$.

The relevance of this to the theory of elliptic operators is twofold. First of all, if E^*, d is an elliptic complex over a smooth compact manifold M, the bundles π^*E^{even} and π^*E^{odd} on T^*M are isomorphic (via the symbol map) away from the zero section, and therefore the formal difference $[\pi^*E^{\text{even}}] - [\pi^*E^{\text{odd}}]$ is a well-defined element of $K_c(T^*M)$.

Second of all, as with cohomology, for any 'K-oriented' bundle V over a compact oriented manifold M there is a *Thom isomorphism* $K(M) \to K_c(V)$ given by multiplication with a (K-theoretic) 'Euler class', i.e. the unique element of $K_c(V)$ whose restriction to each fiber V_p is the generator of $K_c(V_p) \cong \mathbb{Z}$. A sufficient (but not necessary) condition for V to be

K-oriented is that V is (almost) complex, since $K_c(\mathbb{C}^n) \cong \mathbb{Z}$ is essentially the statement of Bott periodicity.

Now if M is a smooth manifold, we can smoothly embed M in $\mathbb{R}^m$ for some sufficiently large m. Let ν denote the normal bundle of this embedding. Then $T\nu$ has a natural (almost) complex structure as a bundle over TM (in fact, it is isomorphic to the pullback of $\nu \otimes \mathbb{C}$ from M to TM). There is a Thom isomorphism $\varphi : K_c(TM) \to K_c(T\nu)$. Identifying ν with a tubular neighborhood of M in $\mathbb{R}^m$ identifies $T\nu$ with an open subset of $T\mathbb{R}^m$ so that we get an inclusion homomorphism[6] $K_c(T\nu) \to K_c(T\mathbb{R}^m) \cong \mathbb{Z}$. The *topological index* of an elliptic complex E^*, d is the image of the class $[\pi^* E^{\text{even}}] - [\pi^* E^{\text{odd}}]$ in $\mathbb{Z}$.

The *Atiyah–Singer Index Theorem*, in the language of K theory, is simply the statement that the analytic index is equal to the topological index.

2.5.2. *Chern character.* The Whitney product formula shows that taking the total Chern class $E \to c(E) := \sum_j c_j(E)$ is a homomorphism from $K(X)$ as an *additive group* to (even dimensional) cohomology $H^{\text{even}}(X)$ as a *multiplicative group*. Atiyah–Hirzebruch showed that over $\mathbb{Q}$ there is a ring homomorphism called the *Chern character* which induces an *isomorphism*:

$$\text{ch} : K(X) \otimes \mathbb{Q} \to H^{\text{even}}(X; \mathbb{Q})$$

The homomorphism property more or less determines ch uniquely. Over X, the set of line bundles is a group with respect to $\otimes$, and the map $L \to c_1(L)$ is the unique homomorphism (up to scale) to H^2 with its *additive* structure. So for a line bundle L we should have

$$\text{ch}(L) = e^{c_1(L)} = \sum_j \frac{c_1(L)^j}{j!}$$

Likewise, for a sum of line bundles $E = \oplus L_k$ we should have

$$\text{ch}(E) = \sum_k e^{c_1(L_k)} = \text{rk}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots$$

where we express the result purely in terms of the Chern classes of E. By the splitting principle this formula defines ch in general.

On a smooth compact manifold M, a complex vector bundle E of rank n can be given a Hermitian metric, reducing its structure group to $\text{U}(n)$. A unitary connection ∇ has a curvature Ω, which is a 2-form with coefficients in the Lie group of $\text{U}(n)$; i.e. in local coordinates, Ω is given by a skew-Hermitian matrix of 2-forms. The Chern classes as differential forms are given (locally) as the coefficients of the characteristic polynomial of the matrix $i\Omega/2\pi$; i.e. if x_j denote the eigenvalues[7] of the matrix locally, then, at least where these eigenvalues are distinct, E splits as a sum of line bundles L_j with $c_1(L_j)$ given by the forms x_j.

In this language, the Chern class $c_k(E)$ is expressed as the kth elementary symmetric polynomial in the eigenvalues:

$$c_k(E) = \sum_{|I|=k} (-1)^k x_{i_1} \cdots x_{i_k}$$

[6] a compactly supported class on an open submanifold can be canonically extended by zero

[7] note the x_j are closed 2-forms on M

and

$$\mathrm{ch}(E) = \sum_k \frac{1}{k!}(x_1^k + \cdots + x_n^k)$$

Equivalently, $\mathrm{ch}(E) = \mathrm{tr}\exp(i\Omega/2\pi)$.

2.5.3. *Todd class and the topological index.* Let X be a compact topological space, and V a complex vector bundle over X. Then V is oriented in the sense of both cohomology and K theory, and there are Thom isomorphisms $\varphi : K(X) \to K_c(V)$ and $\psi : H^*(M) \to H_c^*(V)$, each given by multiplication by the respective 'Euler class' of the two theories.

The Chern character defines natural isomorphisms over $\mathbb{Q}$ between K theory and even dimensional cohomology. However, these isomorphisms do *not* commute with the two Thom isomorphisms. There is a universal correction term, which can be expressed in terms of the Chern classes of V, and which we write $\mu(V)$:

$$\psi^{-1}\mathrm{ch}\,\varphi(u) = \mathrm{ch}\,u \cdot \mu(V)$$

In terms of the eigenvalue notation from § 2.5.2 we have

$$\mu := \prod \frac{1 - e^{x_j}}{x_j}$$

Recall our construction of the topological index via an embedding $M \to \mathbb{R}^m$ for large m. If νM denotes as before the normal bundle of this embedding, then $\nu M \oplus TM$ is trivial, and therefore the correction term $\mu(\nu \otimes \mathbb{C}) = \mu^{-1}(TM \otimes \mathbb{C})$. Since $TM \otimes \mathbb{C}$ is the complexification of a real bundle, it is isomorphic to its dual. In the eigenvalue notation, dualizing replaces x_j by $-x_j$ so we can write this class as

$$\prod \frac{x_j}{1 - e^{x_j}} = (-1)^n \prod \frac{x_j}{1 - e^{-x_j}}$$

This motivates the definition of the *Todd class* of a complex bundle, by

$$\mathrm{Td}(E) := \prod \frac{x_j}{1 - e^{-x_j}}$$

By the naturality of ch, and the correction term for the Thom isomorphisms, one obtains the more 'usual' statement of the Atiyah–Singer Index theorem:

Theorem 2.7 (Atiyah–Singer Index Theorem)**.** *Let M be a closed smooth manifold of dimension n, and let E^*, d be an elliptic complex. Then the index of d is given by the formula*

$$\textit{index of } d = (-1)^n \int_{TM} \mathrm{ch}([\pi^* E^{even}] - [\pi^* E^{odd}]) \cdot \mathrm{Td}(TM \otimes \mathbb{C})$$

2.5.4. *Computing the Index and the proof of Rochlin's Theorem.* If M is even dimensional and orientable, the Index Theorem simplifies somewhat. We can apply the inverse of the Thom isomorphism in cohomology to reduce the computation to an integral over M.

Let $e(M)$ denote the Euler class of M. A short computation gives

$$\text{index of } d = (-1)^{n/2} \frac{\mathrm{ch}(E^{\text{even}}) - \mathrm{ch}(E^{\text{odd}})}{e(M)} \mathrm{Td}(TM \otimes \mathbb{C})[M]$$

Recall that on an oriented even dimensional n-manifold, the *Euler class* is expressed in eigenvalue notation by the product $x_1 \cdots x_{n/2}$. We have $TM \otimes \mathbb{C} = TM \oplus \overline{TM}$ so

$$\mathrm{Td}(TM \otimes \mathbb{C}) = \prod_{j=1}^{n/2} \frac{-x_j^2}{(1-e^{x_j})(1-e^{-x_j})}$$

It remains to compute $\mathrm{ch}(S^\pm)$ for the two spinor bundles. This is done in Atiyah–Singer, and the result is

$$\mathrm{ch}(S^+) - \mathrm{ch}(S^-) = \prod_{j=1}^{n/2}(e^{x_j/2} - e^{-x_j/2})$$

Putting this together gives

$$\mathrm{index}\, \partial\!\!\!/^+ = (-1)^{n/2} \int_M \prod_{j=1}^{n/2} \frac{x_j/2}{\sinh x_j/2}$$

The right hand side has a canonical expansion as a rational power series in the Pontriagin classes of M, called the $\hat{A}$ *genus*; the first few terms are

$$\hat{A}(M) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - p_2) + \cdots$$

Since on a 4-manifold $\sigma = p_1/3$ we deduce that $\mathrm{index}\, \partial\!\!\!/^+ = -\sigma/8$.

2.5.5. *Twisted Dirac operator.* It will be important in the sequel to consider generalizations of the Dirac operator mapping between sections of $S^\pm$ twisted by a (complex) line bundle L. For such an operator $\partial\!\!\!/_A^+$[8] the index must be corrected by a term coming from L. The only relevant characteristic class is $c_1(L)$, and the Index formula gives:

$$\mathrm{index}\, \partial\!\!\!/_A^+ = -\frac{\sigma}{8} + \frac{1}{2}\int_M c_1(L)^2$$

2.6. **Genus bounds.** Let W be smooth and simply-connected. Recall that an element $\omega \in H^2(W;\mathbb{Z})$ is characteristic if its mod 2 reduction is equal to the second Stiefel-Whitney class $w_2 \in H^2(W;\mathbb{Z}/2\mathbb{Z})$, and therefore satisfies $\omega \cdot x = x \cdot x$ mod 2 for all $x \in H^2(W;\mathbb{Z})$. Furthermore, we have seen that $\omega \cdot \omega = \sigma$ mod 8. We explain how to use Rochlin's Theorem to get lower bounds on the genus of smoothly embedded surfaces representing homology classes in certain W.

Example 2.8. Let $W = \mathbb{CP}^2 \#^8 \overline{\mathbb{CP}}^2$, a complex projective plane blown up at eight points. Let α_0 generate $H^2(\mathbb{CP}^2;\mathbb{Z})$ and let α_i generate the ith copy of $H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. The element

$$\omega := 3\alpha_0 + \alpha_1 + \cdots + \alpha_8$$

is characteristic, and $\omega \cdot \omega = 1$. Now, $\sigma = -7$ so $\omega \cdot \omega - \sigma = 8$.

Suppose ω were Poincaré dual to a smoothly embedded sphere Σ. Since $\omega \cdot \omega = 1$, it would follow that the boundary of the unit normal bundle N of Σ is the Hopf bundle over S^2, and is therefore diffeomorphic to S^3. Thus we could do a surgery to define $W' := W - N \cup B^4$.

[8] the A here denotes a choice of connection on L; more on this later

Since Σ is the Poincaré dual of a characteristic element in W, this surgery gives $w_2(W') = 0$ so that W' has even intersection form. On the other hand, by calculation, $\sigma(W') = -8$, contradicting Rochlin's theorem. It follows that the least genus of a smooth embedded surface dual to ω is 1.

This example applies as well to $\mathbb{CP}^2$ itself. If there were a smooth embedded sphere Σ in $\mathbb{CP}^2$ representing 3α, then after blowing up at eight points we would obtain a new sphere Σ' in W as above representing precisely the class ω. Since we have shown this is impossible, it follows that the least genus of a smooth embedded surface in $\mathbb{CP}^2$ representing 3 times the generator is 1.

3. The Thom Conjecture

The purpose of this section is to present the proof, by Kronheimer and Mrowka, of the Thom Conjecture on the minimal genus of smooth embedded representatives of homology classes in $\mathbb{CP}^2$. Their proof uses the new (at the time) invariants discovered by Seiberg and Witten, which arose in the context of certain dualities that are significant in string theory. In these notes we cannot do an adequate job of describing the physical significance of these invariants, but we can (and do) at least gives their definitions, and sketch the proofs of their basic properties.

Theorem 3.1 (Kronheimer–Mrowka, 'The Thom Conjecture'). *Let $d \geq 0$ be an integer. Then the least genus of a smooth embedded surface in $\mathbb{CP}^2$ representing d times the hyperplane class is $(d-1)(d-2)/2$. Any smooth complex degree d curve is therefore a least genus representative.*

3.1. **Gauge theory.**

3.1.1. *Electromagnetic 4-potential.* The fundamental example of a gauge theory is the introduction of the *vector potential* as a convenient device for the solution of the (classical) Maxwell's equations.

On $\mathbb{R}^3$ the Electric and Magnetic fields are expressed as 1-forms

$$E := E_x dx + E_y dy + E_z dz, \quad B := B_x dx + B_y dy + B_z dz$$

but because the symmetries of the Maxwell equations are those of special relativity, it is convenient to work on Minkowski spacetime with the 'metric' $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, and combine E and B together into a two-form

$$F := E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

In coordinate-free notation, $E = -\iota_{\partial_t} F$ and $B = *(F|_{\mathbb{R}^3})$[9]

In this formulation, Maxwell's equations become particularly simple. Gauss' law for magnetism, and Faraday's law for induction are the equations

$$\nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}$$

which together amount to the equation $dF = 0$; i.e. F is *closed*.

[9] Hodge star (i.e. cross product of vectors) is taken on $\mathbb{R}^3$

Similarly, if we combine the electric charge density $\rho := \rho_t dx \wedge dy \wedge dz$ and the electric current density $j := j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy$ into a 3-form [10]

$$J := \rho_t dx \wedge dy \wedge dz + j_x dy \wedge dz \wedge dt + j_y dz \wedge dx \wedge dt + j_z dx \wedge dy \wedge dt$$

then Gauss' law for electricity, and Ampère's circuit law,

$$\nabla \cdot E = \rho, \quad \nabla \times B = j + \frac{\partial E}{\partial t}$$

(in units in which $\epsilon_0 = \mu_0 = 4\pi = c = 1$) together amount to the equation $d * F = J$.

In the absence of sources, these two equations reduce to $dF = d * F = 0$. On Euclidean $\mathbb{R}^4$, this would be equivalent to the condition that F should be *harmonic*. But on Minkowski space, it is equivalent to the *wave equation* $\Box F = 0$ where $\Box := \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ is the *d'Alembertian*, the Minkowski analog of the Laplacian. This is *not* an elliptic equation, and solutions have finite (fixed) propagation speed (as a function of t); this reflects the finiteness and constancy of the speed of light (set to 1 in these coordinates).

Since $dF = 0$, we can write $F = dA$ for some 1-form A, the *electromagnetic 4-potential*, and think of $d * F = J$ as an equation for A. Evidently, the set of solutions is unchanged by transformations of the form $A \to A + df$. Such transformations form a group, the *gauge group* of the equations.

3.1.2. *Electromagnetic potential on curved spacetime.* Let M be a 4-manifold with a Lorentz metric. In this context, the electromagnetic potential A is a *connection* on a complex line bundle L. Covariant differentiation with respect to the connection A is a first-order differential operator $\nabla^A : \Gamma(L) \to \Omega^1(M) \otimes \Gamma(L)$. In local coordinates, where L can be trivialized, we can choose a nonzero section e and write any other section $s = fe$ where f is a smooth (complex-valued) function on M. In this coordinate, there is a complex-valued 1-form θ so that

$$\nabla^A s = dfe + f\theta e$$

If $e' := ge$ is another local trivialization, $s = fg^{-1}e'$. If we write $f' = fg^{-1}$ and let θ' be the 1-form such that $\nabla^A s = df'e' + f'\theta' e'$ then

$$dfe + f\theta e = d(f'g)g^{-1}e' + f'g\theta g^{-1}e' = df'e' + f'(dgg^{-1} + g\theta g^{-1})e'$$

so that we obtain the transformation law $\theta' = dgg^{-1} + g\theta g^{-1}$. Now, $g\theta g^{-1} = \theta$ and $dgg^{-1} = d(\log(g))$ so that the curvature form $d\theta = d\theta' = F$ is well-defined and closed independent of the choice of local coordinate. Thus, $dF = 0$, and the equation $d * F = 0$ (in the absence of charges) is a coordinate-free expression of Maxwell's equations.

If we fix a connection A_0 on L, any other connection A can be expressed in the form $A_0 + a$ for some globally well-defined 1-form a [11] and a gauge transformation sends $a \to a + d(\log(g))$ for g a section of $\text{Aut}(L)$. Thus there is a gauge transformation, unique up to multiplication of g by a nonzero constant, that sends a to a' with $d * a' = 0$. Such an a' is called a *Coulomb gauge* in the physics literature.

[10] possibly J should be the negative of this expression, because we are taking Hodge star in Minkowski space

[11] technically we should think of a as having coefficients in $\text{End}(L)$, but for L a line bundle $\text{End}(L) = L \otimes L^*$ is a trivial line bundle.

3.1.3. *Other gauge groups and Yang–Mills.* Now let E be a vector bundle with structure group G over a 4-manifold M, and let A be a connection on E. Covariant differentiation with respect to A is a first-order operator $\nabla^A : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$. In an open set where E is trivial, if e_i is a basis of local sections of E we can write any section ψ as $\psi = \sum_i \psi_i e_i$ and then there is a matrix of 1-forms A_{ij} so that

$$\nabla^A \psi = \sum_i d\psi_i e_i + \sum_i \psi_i A_{ij} e_j$$

If g is a section of $\mathrm{Aut}(E)$ locally, so that $e_i' = g_{ij} e_j$ is another local trivialization, the matrix of 1-forms transforms like $A' = dgg^{-1} + gAg^{-1}$. Note that at each point the matrix g_{ij} is in G[12] and dgg^{-1} is a 1-form with values in the Lie algebra $\mathfrak{g}$. If parallel transport by A preserves the G structure (i.e. parallel transport around a loop induces an automorphism by an element of G); then A is also given locally by a 1-form with coefficients in $\mathfrak{g}$, and we see that the gauge group preserves this condition.

The structure group acts by $G \to \mathrm{Aut}(E)$ and therefore we also get $\mathfrak{g} \to \mathrm{End}(E)$. There are two natural algebra structures on the fibers of $\mathrm{End}(E)$, namely matrix product and Lie bracket respectively. For p and q forms α, β with coefficients in $\mathrm{End}(E)$ we can first 'multiply' the forms using one of these algebra structures, together with wedge product of (ordinary) forms entrywise; we write these two multiplications as $\alpha \wedge \beta$ and $[\alpha, \beta]$ respectively. These two multiplications are related by

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha$$

Any connection on E determines a connection on E^* and thus on $\mathrm{End}(E)$. The operators d on forms and ∇^A on E thus together determine a connection on $\Omega^k(M, \mathrm{End}(E))$ by

$$\nabla_A \alpha = d\alpha + [A, \alpha]$$

If ψ is any section of E, this satisfies the Leibniz rule

$$\nabla_A(\alpha(\psi)) = (\nabla_A \alpha)\psi + (-1)^k \alpha(\nabla_A \psi)$$

The curvature of the connection is expressed as $F = dA + A \wedge A = dA + \frac{1}{2}[A, A]$. Under a change of coordinates it transforms by $F \to gFg^{-1}$; thus F is well-defined as a 2-form with values in $\mathrm{End}(E)$. We deduce the *Bianchi identity*

$$\nabla^A F = dF + [A, F] = d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A = 0$$

If M has a (Lorentz or Riemannian) metric, it makes sense to take the Hodge dual $*F$ and we have the *Yang–Mills equation*

$$\nabla^A * F = 0$$

In the case $G = \mathrm{U}(1)$ this and the Bianchi identity recovers Maxwell's equations.

3.2. **Seiberg–Witten equations.**

[12] actually g is in the image of G under its representation as the structure group of a fiber of E; the group G need not be a matrix group, as we have already seen.

3.2.1. Spin^c *structures.* Not every closed oriented 4-manifold admits a spin structure; however, it turns out there is a closely related structure that *does* always exist, called a 'spin^c structure'.

Recall that for each n, the Lie group $\mathrm{Spin}(n)$ is the (unique) connected double cover of $\mathrm{SO}(n;\mathbb{R})$. A generalization, $\mathrm{Spin}^c(n)$ is, as a topological space, a connected double cover of $\mathrm{SO}(n;\mathbb{R}) \times \mathrm{U}(1)$, and as a group, is a $\mathbb{Z}/2\mathbb{Z}$ extension which is nontrivial on each factor:

$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n;\mathbb{R}) \times \mathrm{U}(1)$$

Said another way, $\mathrm{Spin}^c(n)$ is a twisted product $\mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}(1)$. Projecting onto $\mathrm{Spin}/\mathbb{Z}/2\mathbb{Z}$ gives us a fibration

$$S^1 \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$$

Lemma 3.2. *On any oriented Riemannian n-manifold M, the set of* Spin^c *structures is in bijection with the set of cohomology classes $e \in H^2(M;\mathbb{Z})$ whose mod 2 reduction is equal to w_2.*

For $n = 4$, such an e always exists, and therefore every oriented Riemannian 4-manifold W admits a Spin^c *structure.*

Proof. There is a fibration $S^1 \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ which extends (e.g. by the homotopy long exact sequence) to

$$S^1 \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \to K(\mathbb{Z},2) \to B\mathrm{Spin}^c(n) \to B\mathrm{SO}(n) \to K(\mathbb{Z};3)$$

The last map is the Bockstein applied to w_2; thus w_2 is in the image of a class $e \in H^2(M;\mathbb{Z})$ if and only if the canonical $\mathrm{SO}(n)$ structure lifts to a Spin^c structure.

To see that such an integral lift exists in dimension 4, we follow an argument from Teichner–Vogt [13]. The coefficient homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induces a map between universal coefficient sequences:

$$\begin{array}{ccccc} \mathrm{Ext}(H_1(W;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^2(W;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}(H_1(W;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^2(W;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \end{array}$$

The map $\mathrm{Ext}(H_1(W;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Ext}(H_1(W;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ is surjective, since Ext^2 vanishes with $\mathbb{Z}$ coefficients. Thus we just need to show that the image of w_2 in $\mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ lifts to $\mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z})$.

Now, the image of the class w_2 in $\mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ is the homomorphism

$$\rho(x) := w_2(x) = x \cdot x \bmod 2$$

for all $x \in H_2(X;\mathbb{Z})$, where $\cdot$ denotes the intersection pairing (i.e. the dual of the cup product pairing) of an *embedded* oriented surface representing the homology class x. Now, because $x \cdot x$ vanishes on torsion, it follows that ρ factors through $H_2(W;\mathbb{Z})/\text{torsion} \to \mathbb{Z}/2\mathbb{Z}$. But $H_2(W;\mathbb{Z})/\text{torsion}$ is free, so ρ lifts to $\mathrm{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z})$. □

Remark 3.3. If W^4 is simply-connected, then

$$H^2(W;\mathbb{Z}) \to H^2(W;\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} = H^2(W;\mathbb{Z}/2\mathbb{Z})$$

is surjective, so *any* class $w \in H^2(W;\mathbb{Z}/2\mathbb{Z})$ has an integral lift.

Example 3.4. If M is a complex manifold (of any dimension) then M is always orientable, and $c_1(M)$ is always an integral lift of w_2. So we can always find a Spin^c structure with $L^2 = K^{-1}$, the 'canonical' bundle.

3.2.2. *Twisted Dirac operator.* In 4 dimensions, the spin group $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathbb{C}^4$, and $\mathrm{Spin}^c(4)$ can be thought of as the subgroup of $\mathrm{U}(2) \times \mathrm{U}(2)$ (which also acts on $\mathbb{C}^4$) consisting of pairs of complex 2×2 matrices of the form λA^+, λA^- where $A^\pm \in \mathrm{SU}(2)$ and $\lambda \in \mathrm{U}(1)$.

Associated to a Spin^c structure there is a line bundle that we write[13] L^2 coming from the homomorphism $\mathrm{Spin}^c \to \mathrm{U}(1)$. If e is an integral lift of w_2, then $e = c_1(L^2)$. We write the $\mathbb{C}^4$ bundle with its Spin^c structure as $S \otimes L$, even though S and L do not necessarily exist as individual bundles, and observe that it decomposes as before into $\mathbb{C}^2$ bundles $S^+ \otimes L$ and $S^- \otimes L$. Note that these bundles still admit the structure of modules over the Clifford algebra of the cotangent bundle, fiberwise, and multiplication by a 1-form interchanges $+$ and $-$.

By abuse of notation, we let $2A$ denote a unitary connection on L^2. In any local trivialization,covariant differentiation of a section of L^2 with respect to any connection has the form $\nabla s = ds + \theta s$ for some pure imaginary 1-form θ, which we think of as a 1-form with values in the Lie algebra of the structure group $\mathrm{U}(1)$ of L^2.

Even though L does not exist globally, the connection on L^2 defines a connection on L locally, since the Lie algebra of the structure group is the same. In local coordinates where $\nabla s = ds + \theta s$ for $s \in \Gamma(L^2)$, we have $\nabla t = dt + \theta/2t$ for $t \in \Gamma(L)$. Again, by abuse of notation, we denote this connection on L by A.

Together, the Levi-Civita connection and $2A$ define a unique connection on $S^\pm \otimes L$, and we denote the covariant derivative

$$\nabla^A : \Gamma(S^\pm \otimes L) \to \Gamma(T^*M \otimes S^\pm \otimes L)$$

Now for ψ a section of $S \otimes L$, define the *Dirac operator*

$$\partial\!\!\!/_A \psi := \sum_i e_i \cdot \nabla_i^A \psi$$

where the sum is taken over an orthonormal basis of 1-forms e_i as before, and define $\partial\!\!\!/_A^\pm$ to be the components mapping $\Gamma(S^\pm \otimes L) \to \Gamma(S^\mp \otimes L)$. As before, one can check that for any fixed A, the operator $\partial\!\!\!/_A^+$ is elliptic, with adjoint $\partial\!\!\!/_A^-$.

If a connection on a line bundle is expressed in terms of a local trivialization as $\nabla s = ds + \theta s$, then the curvature is the 2-form $d\theta$. Note that θ is not globally well-defined as a 1-form; a change of local coordinates changes θ by a closed 1-form (more on this later). But $d\theta$ *is* globally well-defined, and is a closed 2-form. If F_{2A} is the curvature of L^2 for the connection $2A$, then the curvature of L for the connection A is $F_A = 1/2F_{2A}$. This is a well-defined closed 2-form, whose cohomology class equals $c_1(L)/(2\pi i) = e/(4\pi i) \in H^2(W;\mathbb{R})$.

This 2-form decomposes into self-dual and anti self-dual parts $F_A^\pm$.

[13]this is the standard notation, even though L^2 is not necessarily the tensor square of an honest line bundle

3.2.3. *The quadratic form.* If ψ is a section of $S^+ \otimes L$, so that locally we can write $\psi = \binom{\psi_1}{\psi_2}$, then we can define a quadratic form

$$q(\psi) := \text{trace-free part of } \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\begin{pmatrix}\overline{\psi}_1 & \overline{\psi}_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}|\psi_1|^2 - |\psi_2|^2 & 2\psi_1\overline{\psi}_2 \\ 2\psi_2\overline{\psi}_1 & |\psi_2|^2 - |\psi_1|^2\end{pmatrix}$$

Another way to write this is

$$q(\psi) = (\psi \otimes \psi^*) - |\psi|^2 \text{Id}$$

We can think of $q(\psi)$ as a section of the bundle of endomorphisms of $S^+ \otimes L$.

Lemma 3.5. *The endomorphism $q(\psi)$ acts on $S^+ \otimes L$ as Clifford multiplication by a pure imaginary self-dual two-form, and the norm of $q(\psi)$ as a two-form is $|\psi|^2/2\sqrt{2}$.*

Proof. We can check that in our choice of local coordinates, the generators of Λ^2_+ act by

$$e_1 \cdot e_2 + e_3 \cdot e_4 = \begin{pmatrix} -2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_1 \cdot e_3 + e_4 \cdot e_2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$e_1 \cdot e_4 + e_2 \cdot e_3 = \begin{pmatrix} 0 & -2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $q(\psi)$ may be expressed locally in a unique way as an imaginary linear combination of these generators.

Notice further that each of these generators has length $\sqrt{2}$ as a 2-form, and its image has length $2\sqrt{2}$ in the matrix norm[14]. Thus length in the matrix norm is twice the length of the corresponding 2-form. We compute

$$\|q(\psi)\|^2 = \frac{1}{4}(2|\psi_1|^4 + 2|\psi_2|^4 - 4|\psi_1|^2|\psi_2|^2 + 8|\psi_1|^2|\psi_2|^2) = \frac{1}{2}|\psi|^4$$

so that $\|q\| = |\psi|^2/\sqrt{2}$, and the length of the corresponding 2-form is $|\psi|^2/2\sqrt{2}$. □

3.2.4. *The Seiberg–Witten equations.* We now define the Seiberg–Witten equations, for an oriented Riemannian 4-manifold W together with a choice of line bundle L^2 with $c_1(L^2) = e$ congruent to w_2 mod 2. These are equations for a pair (A, ψ) where A is a connection 'on L', and ψ is a section of $\Gamma(S^+ \otimes L)$. With notation as in the previous few sections, the equations are:

$$\not{\partial}^+_A \psi = 0$$

and

$$F^+_A = q(\psi)$$

It is important also to consider a *perturbation* of these equations, for reasons that shall become clear in the sequel. In the perturbed equations, we fix a smooth closed pure imaginary self-dual two-form ϕ, and replace the second equation above with

$$F^+_A = q(\psi) + \phi$$

[14] i.e. for a matrix M, we have $\|M\|^2 := \sum |M_{ij}|^2$

These equations are very mildly nonlinear: the only nonlinearity resides in the quadratic term $q(\psi)$.

Remark 3.6. There are several different conventions in the literature, most of which have to do with different choices of normalizations for F_A and q. In the physics literature, the connection 1-form and the curvature of a unitary line bundle is real; see [14], p. 772.

3.2.5. *Gauge invariance.* Let's denote by $\mathcal{A}$ the space of pairs (A, ψ) where A is a connection 'on L' and ψ is a section of $S^+ \otimes L$. The Seiberg–Witten equations pick out a subset of $\mathcal{A}$, the space of solutions to the equations, which we denote $\mathcal{B}$. We likewise denote by $\mathcal{B}_\phi$ the set of solutions to the perturbed equations associated to a self-dual imaginary 2-form ϕ.

One key difference between the Seiberg–Witten and the 'ordinary' Dirac equations is the existence of an infinite dimensional family of *gauge symmetries* acting on $\mathcal{A}$ and preserving the subspace $\mathcal{B}$.

The key point is that the only essential way that the connection A enters into the equations is via its curvature. The fibers of a $U(1)$ bundle are modules for the action of $U(1) = S^1$, and the group of smooth maps $\text{Map}(W, S^1)$ acts on a $U(1)$ bundle by automorphisms. One should think of this action as something inessential: it is really just a change of coordinates on a fixed underlying 'geometric' object. Two connections on the same bundle are isomorphic under a bundle automorphism if and only if they have the same curvature. Parallel sections with respect to one connection are taken to parallel connections with respect to the other.

Let's fix a $U(1)$ bundle L. With respect to a local trivialization, the covariant derivative ∇ associated to a connection A acts on a section s by $\nabla s = ds + ias$ for some smooth 1-form a. This expression is not tensorial, but if ∇_0, ∇_1 are the covariants derivatives associated to *two* connections A_0, A_1 their *difference* is of the form $-ia$ for some honest smooth 1-form a, invariantly defined. Thus, the space of connections on L is an affine space for $i\Omega(W)$, and if we fix a basepoint A_0, any A may be represented uniquely as $A_0 - ia$.

Every fiber of L admits a canonical $U(1)$ action; thus the group of smooth maps $\text{Map}(W, S^1)$ acts by automorphisms of L. We call $\mathcal{G} := \text{Map}(W, S^1)$ the *gauge group*. An automorphism $g \in \mathcal{G}$ transforms a covariant derivative, expressed in local coordinates as $s \to ds + ias$, to

$$s \to gd(g^{-1}s) + gia(g^{-1}(s)) = ds + (gdg^{-1})s + ias$$

where we use the fact that the Lie algebra of $U(1)$ is abelian. Thus the action on the space of connections is given by

$$A_0 - ia \to A_0 - ia + gdg^{-1}$$

Likewise, a section ψ of $S^\pm \otimes L$ transforms under $g \in \mathcal{G}$ by $\psi \to g\psi$.

If W is simply-connected, any $g \in \mathcal{G}$ can be written as $g = e^{iu}$ for some real-valued function u, unique up to adding an element of $2\pi\mathbb{Z}$, and therefore the gauge group $\mathcal{G}$ acts on pairs $(A_0 - ia, \psi)$ by

$$(A_0 - ia, \psi) \to (A_0 - i(a + du), e^{iu}\psi)$$

Lemma 3.7. *The group $\mathcal{G}$ preserves the subspaces $\mathcal{B}$ and $\mathcal{B}_\phi$ for any ϕ.*

Proof. Since U(1) is abelian, the forms F_A and F_A^+ don't change under the gauge group. We have

$$q(e^{iu}\psi) = ((e^{iu}\psi) \otimes (e^{-iu}\psi^*)) - |\psi|^2\mathrm{Id} = (\psi \otimes \psi^*) - |\psi|^2\mathrm{Id} = q(\psi)$$

so the action of $\mathcal{G}$ preserves solutions to the second SW equation.

Writing ψ locally as $\phi \otimes s$ for sections ϕ of S^+ and s of L we have

$$\partial\!\!\!/_A\psi = \sum_i e_i \cdot \nabla_i^A \psi = \sum_i e_i \cdot (\nabla\phi \otimes s + \phi \otimes (ds - ias))(\partial_i)$$

and therefore if $A' = A - idu$,

$$\partial\!\!\!/_{A'} e^{iu}\psi = \sum_i e_i \cdot (\nabla_i \phi \otimes e^{iu}s + \phi \otimes (\partial_i(e^{iu}s) - iae^{iu}s - idue^{iu}s))(\partial_i) = e^{iu}\partial\!\!\!/_A\psi$$

so the action of $\mathcal{G}$ preserves solutions to the first SW equation, and the lemma is proved. □

Let $\mathcal{G}_0$ denote the subgroup of $\mathcal{G}$ consisting of functions $g = e^{iu}$ with $\int_W u = 0$. This subgroup is normal, and there is a split exact sequence

$$\mathcal{G}_0 \to \mathcal{G} \to S^1$$

where S^1 is represented by the constant maps $W \to S^1$.

Proposition 3.8. *The group $\mathcal{G}_0$ acts freely on the space $\mathcal{A}$ of pairs $(A_0 - ia, \psi)$. Under this action, each orbit contains a unique representative with $\delta a = 0$.*

Proof. Evidently, if $g \in \mathcal{G}_0$ acts trivially then $du = 0$ so u is a constant. Since $\int_W u = 0$ we have $u = 0$ and $g = \mathrm{id}$.

Now, by the Hodge theorem, we have $\Omega^1 = \ker\delta \oplus d\Omega^0$, so for any $-ia'$ there is a unique du with $\delta(-ia' + du) = 0$. □

The group S^1 acts freely on the quotient $\mathcal{A}/\mathcal{G}_0$ (and therefore all of $\mathcal{G}$ acts freely on $\mathcal{A}$) *except* at points of the form $(A - ia, 0)$; these points are called *reducible.* A reducible element of $\mathcal{B}$ is likewise called a *reducible solution.*

We form the following quotients

$$\tilde{\mathcal{M}} := \mathcal{B}/\mathcal{G}_0 \text{ and } \mathcal{M} := \mathcal{B}/\mathcal{G}$$

and call $\mathcal{M}$ the *moduli space of solutions* to the SW equations. Likewise we define $\tilde{\mathcal{M}}_\phi = \mathcal{B}_\phi/\mathcal{G}_0$ and $\mathcal{M}_\phi = \mathcal{B}_\phi/\mathcal{G}$.

Proposition 3.9. *Let W be a smooth, oriented, closed Riemannian 4-manifold. If $b_2^+ > 0$ then for a generic self-dual imaginary 2-form ϕ the space $\mathcal{B}_\phi$ contains no reducible solutions, and consequently the moduli space $\mathcal{M}_\phi$ is a* smooth oriented manifold *whose dimension is*

$$\dim(\mathcal{M}_\phi) = 2(\textit{index of } \partial\!\!\!/_A^+) - b_2^+ - 1 = c_1(L)^2[W] - \frac{1}{4}(2\chi(W) + 3\sigma(W))$$

If $b_2^+ > 1$ then the space of generic self-dual imaginary 2-forms for which $\mathcal{B}_\phi$ contains no reducible solutions is connected.

Proof. We must first complete the various spaces to Banach manifolds so that we may apply Sard-Smale. Then we must linearize the SW equations at a (non-reducible) solution, and compute the dimension of the tangent space to $\mathcal{M}_\phi$ using Atiyah-Singer.

The first equation is already linear, and the index is $-\sigma/8 + c_1(L)^2[W]/2$, as we saw in § 2.5.5. Of course, this is the index as a difference of dimensions of *complex* vector spaces; the contribution to the dimension of $\mathcal{M}_\phi$ as a real manifold is $-\sigma/4 + c_1(L)^2[W]$.

The linearization of the second equation has an index which can be computed from the following complex:

$$0 \to \Omega^0(W) \xrightarrow{d} \Omega^1(W) \xrightarrow{d^+} \Omega^2_+(W) \to 0$$

where d^+ is the composition of d with projection to the self-dual 2-forms. Roughly, the image of the first d parameterizes the tangent space to the action of the gauge group, while the second d^+ parameterizes the different choices of F_A^+ obtained by varying the connection A.

In more detail: if we fix a basepoint A_0 in the space of connections, with curvature F_{A_0}, then as A ranges over the space of connections, its curvature ranges over $F_A = F_{A_0} + ida$ for $a \in \Omega^1(W)$ (in other words, the possible values of $i/(2\pi)F_A$ range precisely over representatives of the de Rham cohomology class $[i/(2\pi)F_A] = c_1(L) = c_1(L^2)/2$).

If W is simply-connected (or if we merely have $b_1 = 0$) the index of this complex is $-b_2^+ - 1$, which can be interpreted as the (negative of) the dimension of the space of 'constraints' imposed by the linearization of the second equation. Putting this together gives the desired formula for the dimension of $\mathcal{M}_\phi$ (assuming no reducible solutions).

Finally we consider the conditions on ϕ under which there is a reducible solution $(A, 0)$. Let $\mathcal{H}^2_+$ denote the space of harmonic self-dual 2-forms; as we have seen, the dimension of this space is b_2^+. The Hodge theorem says that there is an orthogonal decomposition $\Omega^2_+ = \mathcal{H}^2_+ + d^+\Omega^1$, so there is a reducible solution if and only if the projection of ϕ to $i\mathcal{H}^2_+$ is equal to the self-dual part of the unique harmonic representative of $[F_A]$. When $b_2^+ > 0$ the complement $i\mathcal{H}^2_+ - F_A^+$ is open and nonempty; when $b_2^+ > 1$ it is connected. □

Example 3.10. If X is complex (or more generally, almost-complex) there is a 'canonical' Spin^c structure with

$$c_1(L) = c_1(L^2)/2 = c_1(X)/2, \quad c_2(X) = \chi(X), \text{ and } c_1(L^2)^2 - 2c_2 = p_1 = 3\sigma$$

Thus

$$\dim(\mathcal{M}_\phi) = \frac{1}{4}(c_1(L^2)^2 - 2c_2 - p_1) = 0$$

In particular, $\mathcal{M}_\phi$ is a finite set of points, whose (signed) count is an invariant, at least when $b_2^+ > 1$.

3.2.6. *The magic of Weitzenböck formulas.* The title of this subsection comes from the paper [1] by J.-P. Bourguignon. Briefly, a Weitzenböck formula is an algebraic identity between natural differential 2nd order operators on a Riemannian manifold.

Let E be a 'natural' bundle on a Riemannian manifold M (e.g. some bundle obtained functorially from the principle $O(n)$-bundle associated to the metric).

Definition 3.11. A second-order differential operator $A : \Gamma(E) \to \Gamma(E)$ is called a *coarse Laplacian* if its symbol $\sigma(A)$ satisfies

$$\sigma(A)(\theta, \theta)e = -g(\theta, \theta)e$$

for $\theta \in T^*M$ and $e \in E$.

Now, suppose Δ_0, Δ_1 are two coarse Laplacians acting on $\Gamma(E)$. Since they have the same symbol, their difference is a priori a *first order* natural differential operator, whose symbol is therefore an $\mathrm{O}(n)$-invariant map from $T^*M \otimes E$ to E. But by classical invariant theory, the only such map is the zero map;[15] in other words, $\Delta_0 - \Delta_1$ is a 0th order differential operator, which is to say a *tensor field* R. The only natural tensor fields on a Riemannian manifold are those arising from the curvature of M and of E; in other words, we obtain an identity of the form

$$\Delta_0 = \Delta_1 + R$$

where R is some piece of the curvature of M and E.

There is always at least one natural coarse Laplacian, namely $\nabla^*\nabla$, where ∇ is the (metric) covariant derivative on the bundle E, and ∇^* is the formal adjoint in the space of L^2 sections. One then has, for any section e of E, the identity

$$\langle \Delta_0 e, e \rangle = \langle \nabla e, \nabla e \rangle + \langle Re, e \rangle$$

Since $\langle \nabla e, \nabla e \rangle = \|\nabla e\|^2 \geq 0$, it follows that when R is a strictly positive operator, the kernel of Δ_0 is trivial; and when R is non-negative, the kernel of Δ_0 consists of parallel sections.

The simpler the operator R is, the more powerful this observation becomes. The case of the Dirac operator is known as the *Lichnerowicz formula*:

$$\not{\partial}^2 = \nabla^*\nabla + \frac{s}{4}$$

where s is the scalar curvature. Note that this immediately implies that if the scalar curvature of a spin manifold W is strictly positive, there are no harmonic spinors, and therefore (by the index theorem) $\sigma = 0$.

Example 3.12. The manifold $\mathbb{CP}^2$ has a metric of pinched positive curvature, but $\sigma = 1$. This shows the importance of the spin condition.

Let Σ be a closed oriented surface. Then $\Sigma \times S^2$ is spin, and (by shrinking the S^2 factors) has a metric of positive scalar curvature. Thus $\sigma = 0$ (which is easy to see anyway, by the Künneth formula).

The Weitzenböck formula associated to $\not{\partial}_A^+$ has the form

$$(\not{\partial}_A^+)^* \not{\partial}_A^+ = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2} F_A^+$$

where we think of F_A^+ acting by Clifford multiplication on sections of $S^+ \otimes L$.

Lemma 3.13. *Any solution (A, ψ) to the Seiberg–Witten equations satisfies*

$$|\psi|^2 \leq \max(0, -s)$$

[15] this follows from the representation theory of $\mathrm{O}(n)$

Proof. At a maximum for $|\psi|^2$ we have $\Delta|\psi|^2 \geq 0$ where Δ is the usual metric Laplacian on functions. Since ∇_A is a metric connection,

$$0 \leq \Delta|\psi|^2 = 2\langle \nabla_A^* \nabla_A \psi, \psi \rangle - 2\langle \nabla_A \psi, \nabla_A \psi \rangle \leq 2\langle \nabla_A^* \nabla_A \psi, \psi \rangle$$

Now, by the Weitzenböck formula, the right hand side is equal to

$$\text{RHS} = -\frac{s}{2}|\psi|^2 - \langle q(\psi)\psi, \psi \rangle = -\frac{s}{2}|\psi|^2 - \frac{1}{2}|\psi|^4$$

If ψ is nonzero, dividing by $|\psi|^2$ at the maximum proves the lemma. □

This Lemma gives an *a priori* pointwise bound on $|\psi^2|$. Since $|F_A^+| = |\psi|^2/2\sqrt{2}$, we get a pointwise bound on $|F_A^+|$.

Now, $F_A = F_A^+ + F_A^-$, and

$$c_1(L)^2[W] = \frac{-1}{4\pi^2} \int_W F_A \wedge F_A = \frac{1}{4\pi^2} \int_W |F_A^+|^2 - |F_A^-|^2$$

Since the dimension of $\mathcal{M}$ is $c_1(L)^2[W] - 1/4(2\chi(W) + 3\sigma(W))$ which is non-negative if $\mathcal{M}$ is nonempty, we get an *a priori* bound on the L^2 norm of F_A^-.

By means of this lemma and a standard bootstrapping argument, one can show that any sequence of solutions of the Seiberg–Witten equations has a subsequence which converges (after suitable gauge transformation) in C^∞, and therefore the moduli spaces $\mathcal{M}$ and $\mathcal{M}_\phi$ are *compact*, and are empty for all but finitely many choices of Spin^c structure on W. Finally, we note that if W has a metric of positive scalar curvature, the moduli space $\mathcal{M}$ is empty.

3.2.7. *Dependence on the metric and wall-crossing.* We have seen in Proposition 3.9 that for a given metric, the space of ϕ for which $\mathcal{B}_\phi$ has no reducible solution has real codimension b_2^+. If $b_2^+ > 1$ we may join any two metrics by a smooth path, and any two generic choices of ϕ by a smooth path, so that the family of moduli spaces gives a smooth cobordism between the two ends. Thus, if for example the formal dimension of $\mathcal{M}_\phi$ is zero, the number of points in $\mathcal{M}_\phi$ (counted with sign) is an invariant of the Spin^c class.

If $b_2^+ = 1$ there is a *wall-crossing formula*, a correction term which corrects the signed count when we pass from one component of the space of generic ϕ to another. Ignoring signs, it turns out that the parity of $|\mathcal{M}_\phi|$ changes by exactly 1 when we do a wall-crossing. More generally, at a (generic) point on the wall, when the dimension d is even, the moduli space $\mathcal{M}_\phi$ acquires a singularity locally modeled on the cone on a complex projective space $\mathbb{CP}^{d/2}$, and when we pass through a wall, the moduli space changes (up to cobordism) by disjoint union with a $\mathbb{CP}^{d/2}$.

3.3. **Proof of the Thom Conjecture.** In this section we outline the key steps of the proof of the Thom Conjecture, following Kronheimer–Mrowka [7]. For details, one should consult their paper, which is short and lucid.

3.3.1. *Blow-up and choice of Spin^c structure.* For any n, the result of blowing up $\mathbb{CP}^2$ at n points is a complex manifold diffeomorphic to $X := \mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$. Let H be the Poincaré dual of the hyperplane class in $\mathbb{CP}^2$, and E_i the corresponding classes in the $\overline{\mathbb{CP}}^2$ factors; these form a basis for $H^2(X;\mathbb{R})$. The intersection form has signature $(1,n)$, and we let C^+ denote the component of the open positive cone in H^2 containing the class H. Since $b_2^+ = 1$, for every metric g there is a self-dual harmonic form ω_g in C^+ unique up to scaling.

Since X is complex, there is a natural choice of a Spin^c structure, corresponding to a line bundle L^2 with $c_1(L^2) = c_1(X) = w_2$ mod 2. In cohomology, we have $c_1(L^2) = 3H - \sum E_i$. For this choice, the formal dimension of $\mathcal{M}_\phi$ is zero, so that the moduli space consists of a finite number of points, which we count mod 2.

3.3.2. *Good metrics and number of solutions mod 2.*

Definition 3.14. A metric is *good* if the image of $c_1(L)$ in $H^2(X;\mathbb{R})$ is not represented by an anti self-dual form.

If a metric is good, then $F_A^+ + i\phi$ is never zero when $|\phi| \ll 1$; i.e. there are no reducible solutions for small ϕ. Observe that a metric is good if and only if $c_1(L) \cup [\omega_g] \neq 0$.

The following is Proposition 6 in [7]:

Proposition 3.15. *The number of solutions for a good metric g and $|\phi| \ll 1$ is even if $c_1(L) \cup [\omega_g]$ is positive, and odd if $c_1(L) \cup [\omega_g]$ is negative.*

Proof. We have already asserted that the parity changes when we pass transversely through a wall, which happens when $c_1(L) \cup [\omega_g] = 0$, so it suffices to show that the moduli space is empty for some metric with $c_1(L) \cup [\omega_g] > 0$. But it is known that X admits a Kähler metric of positive scalar curvature. By the Weitzenböck formula, $\mathcal{M}$ is empty for such a metric. Moreover, for such a metric ω_g is a positive multiple of the Kähler form, and $c_1(L) \cup [\omega_g]$ is positive for algebraic reasons. ∎

The conclusion is that if g is a good metric with $c_1(L) \cup [\omega_g]$ *negative*, the moduli space $\mathcal{M}_\phi$ is nonempty for $|\phi| \ll 1$.

3.3.3. *Stretching the neck.* Now, let Σ be a smoothly embedded oriented surface representing dH in $\mathbb{CP}^2$. We may assume $d > 3$, and it is known in this case that Σ is not a sphere. After blowing up d^2 times, and taking S_i to be a sphere in the ith copy of $\overline{\mathbb{CP}}^2$ dual to $-E_i$, we let $\tilde{\Sigma}$ be the result of connect summing Σ with all the S_i. This surface has the same genus as Σ, and as a homology class is dual to $dH - \sum E_i$, and therefore has self-intersection number 0 and trivial normal bundle.

Let Y be the boundary of a tubular neighborhood of $\tilde{\Sigma}$, so that $Y = \tilde{\Sigma} \times S^1$, and 'stretch' the metric in a neighborhood of Y so that it becomes isometric to a product $Y \times [-t,t]$ where the metric on Y is itself a product $\tilde{\Sigma} \times S^1$ where $\tilde{\Sigma}$ has constant scalar curvature $-2\pi(4g-4)$ (and therefore area$(\tilde{\Sigma}) = 1$).

If we normalize the forms ω_g in this sequence of metrics so that $H \cup [\omega_g] = 1$ then, since these forms are uniformly bounded in L^2, they converge uniformly to zero in compact regions of the neck; i.e. $[\omega_g] \cap [\tilde{\Sigma}] \to 0$. But then

$$[\omega_g] \cup c_1(L^2) = [\omega_g] \cap [\tilde{\Sigma}] - (d-3)[\omega_i] \cup H \to (d-3)$$

which is negative by hypothesis.

It follows that when t is big enough, the moduli space $\mathcal{M}_\phi$ is nonempty.

3.3.4. *Extracting a convergent subsequence.* For $t \to \infty$ we construct a sequence of product metrics on $Y \times [-t, t]$ as above, and observe that for sufficiently large t the moduli space is nonempty. Kronheimer–Mrowka show ([7], Prop. 8) that some subsequence of the solutions converge on compact subsets of $Y \times \mathbb{R}$ to a *translation invariant* solution. They do this by extracting a suitable quantity C (a kind of transgression class) on the constant t slices, that controls the variation of a solution with respect to the t coordinate, and is monotone as a function of t. An *a priori* bound for this quantity proves the claim.

Now, a translation invariant solution (A, ψ) achieves its supremum (since Y is compact), so we still get the pointwise bound $|\psi|^2 \le 2\pi(4g-4)$ and consequently $|F_A^+| \le \sqrt{2}\pi(2g-2)$.

Since the solution is translation invariant, the form F_A is obtained by pullback from a form on Y, so $|F_A^+| = |F_A^-|$ and $|F_A| \le 2\pi(2g-2)$. But then

$$d^2 - 3d = |c_1(L^2) \cap [\tilde{\Sigma}]| = \left| \frac{i}{2\pi} \int_{\tilde{\Sigma}} F_A \right| \le 2g - 2$$

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