Geom/Top: Homework 1 (due Monday, 10/08/12)

1. Read Farb notes.
2. Read Hatcher, Intro to Chapter 2 and Section 2.1.
3. (Do not hand this in) Let $\phi : A \to B$ and $\psi : B \to A$ be group homomorphisms. Suppose that $\phi \circ \psi = \mathrm{Id}_B$. Prove that ϕ is surjective and ψ is injective.

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1. Give a Δ-complex structure and use it to compute the homology of the following spaces.

   (a) The annulus/cylinder $S^1 \times [0, 1]$.

   (b) The Mobius band.

   (c) $\mathbf{RP}^2$.

   (d) The closed, connected (oriented) surface Σ_2 of genus 2.

   Remark. Computing for any $g \geq 2$ the simplicial homology of the closed surface Σ_g of genus g is possible, but it is rather cumbersome. We will later have various methods for which that computation is very easy.

2. (**Reduced homology groups**) For various formulas in topology and homological algebra, it is convenient to use a slight variation of homology. Let X be any nonempty Δ-complex. The **reduced complex of simplicial chains** on X is the chain complex $\tilde{\mathcal{C}} := \{C_n(X), \partial_n\}$, where we set $C_{-1}(X) = \mathbf{Z}$, we define $\partial_{-1} := 0$, and we define $\partial_0 : C_0(X) \to C_{-1}(X) = \mathbf{Z}$ by

$$\partial_0(\sum_i a_i v_i) := \sum_i a_i$$

   This is in contrast to the (unreduced) complex chain complex on X, where we defined $C_{-1}(X) := 0$. Thus the complex of reduced simplicial chains on X is

$$\cdots \xrightarrow{\partial_4} C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbf{Z} \to 0$$

   Define the **reduced homology $\tilde{H}_i(X)$ of** X by

$$\tilde{H}_i(X) := H_i(\tilde{\mathcal{C}})$$

   (a) Let X be any nonempty Δ-complex. Prove that

$$H_0(X) \approx \tilde{H}_0(X) \oplus \mathbf{Z}$$

   and that $\tilde{H}_i(X) \approx H_i(X)$ for all $i > 0$. In particular, the effect of using reduced homology of a space is that connected spaces X have $\tilde{H}_0(X) = 0$ instead of $H_0(X) \approx \mathbf{Z}$.

(b) Here is an example of a formula that is cleaner using reduced homology. Let X and Y be connected Δ-complexes. Pick $x \in X$ and $y \in Y$. Denote the **wedge** $X \vee Y$ to be the quotient space given by the quotient of the disjoint union of X and Y by the relation $x \sim y$. Prove that

$$\tilde{H}_i(X \vee Y) = \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$$

for all $i \geq 0$.

3. (**Simplicial Maps**) Let X and Y be Δ-complexes. A map $f : X \to Y$ is called a **simplicial map** if it maps simplices to simplices (perhaps of lower dimension): for each simplex $\sigma : [v_0 \cdots v_n] \to X$, there is a simplex $\tau : [v_{i_0} \cdots v_{i_m}] \to Y, 0 \leq m \leq n$ so that $f \circ \sigma$ is the composition $\tau \circ L$, where $L : [v_0 \cdots v_n] \to [v_{i_0} \cdots v_{i_m}]$ is the linear ("collapse") map of $[v_0 \cdots v_n]$ that projects it onto its subsimplex $[v_{i_0} \cdots v_{i_m}]$, taking each v_{i+0} to itself.

   (a) Fix $n \geq 0$. For any n-simplex σ, we have $f(\sigma)$ is an m-simplex for some $m \leq n$. Define $f_{\#}(\sigma)$ to be 0 if $m < n$, and to be the element $f(\sigma) \in C_n(X)$ otherwise. $f_{\#}$ has a unique linear extension $f_{\#} : C_n(X) \to C_n(Y)$. Prove that $f_{\#}$ is a chain map, and so induces homomorphisms $f_* : H_n(X) \to H_n(Y)$. Note that by the general "chain maps" theorem in homological algebra, the association $f \mapsto f_*$ is functorial.

   (d) Deduce that a simplicial homeomorphism $f : X \to Y$ induces an isomorphism of simplicial homology groups.

4. (**Simplicial complexes**) Let X be a Δ-complex, and assume (as usual) that every subsimplex of a simplex in X is a simplex in X. Prove that the following are equivalent:

   (a) Each simplex of X is uniquely determined by its vertices.

   (b) The intersection of any two simplices in X is a subsimplex of each.

   A Δ-complex satisfying any (hence all) of these properties is called a **simplicial complex**.

5. Prove that any Δ-complex is homeomorphic to a simplicial complex.

6. (a) Find a simplicial complex X that is homeomorphic to the torus, using as few 2-simplices as you can.

   (b) Use Euler's Formula $V - E + F = 0$ for the Torus and the definition of simplicial complex to prove your number is minimal. Note that the minimal number is much bigger than the two 2-simplices required for the Δ-complex structure on T^2 given in class.

   (c) Do the same for $\mathbf{RP}^2$.

7. (**Infinite Δ-complexes and simplicial complexes**) The definition of a (not necessarily finite) Δ-complex X is the same as that for finite Δ-complexes except now the set of simplices $\{f_\alpha : \Delta_\alpha \to X\}_{\alpha \in I}$ is indexed

by a set I of arbitrary cardinality, and one must specify a topology. We will endow X with the **weak topology**, where a subset $Y \subseteq X$ is declared to be open if and only if $f_\alpha^{-1}(Y)$ is open in Δ_α for every $\alpha \in I$. One can define infinite simplicial complexes in the same way.

Now let X be a Δ complex.

(a) Prove that X is compact if and only if it has finitely many simplices.

(b) Let Y be any topological space. Prove that any map $h : X \to Y$ is continuous if and only if for every simplex $\sigma : \Delta_n \to X$, the composition $h \circ \sigma$ is continuous.

(c) Suppose that X has countably many simplices, is locally finite and that every simplex of X has dimension at most $n < \infty$. Prove that X can be embedded in $\mathbf{R}^{2n+1}$. Prove that $2n+1$ is sharp.

Extra credit problems

1. Let $f(g)$ be the minimal number of triangles in any triangulation of the closed, orientable surface of genus g. Find $f(g)$. The correct lower bound is elementary (although the answer is rather strange-looking), and can be deduced from obvious relations among V, E and F, together with Euler's formula $V - E + F = 2 - 2g$. The correct upper bound, where one needs "only" construct an efficient enough triangulation, , is significantly more difficult, and wasn't known until the 1960's. It would be nice to find a simpler proof. Also, is there an easy way to see that f is a monotone increasing function of g?

2. Research problem: Let $h(g, n)$ be the number of triangulations of the closed, orientable surface S_g of genus g which have exactly n triangles. Here we consider two triangulations to be the same if there is a homeomorphism of S_g taking one triangulation to the other. Describe $h(g, n)$ as much as possible. How does it grow as a function of g and n as one is fixed and the other goes to infinity? Compute h for small values of g and n.

3. Do triangulations with greater or fewer triangles have bigger or smaller automorphism group? Can you bound the order of the automorphism group of any triangulation on a closed surface of genus g, i.e. can you give a bound depending only on g and not on the triangulation?

4. Research Problem: Let S_g be the (connected, compact, boundaryless) genus $g \geq 0$ surface. For any triangulation T of S_g, we can get a new triangulation T' by performing an *elementary move*: pick an edge of T, remove it so that you get a quadralateral consisting of the union of the

two triangles intersecting at your edge, then add back the other diagonal to this quadrilateral. This clearly gives a triangulation T' of S_g.

Let $\Gamma_g(n)$ be the graph whose vertices are the triangulations of S_g and where two vertices T, T' are connected by an edge when they differ by an elementary move.

GENERAL PROBLEM: What does $\Gamma_g(n)$ look like as a function of g and n? For example, $\Gamma_g(n)$ is empty whenever $n < f(g)$, where f is the function defined in Problem 2 above. Here are some sample problems/questions:

- Try to determine $\Gamma_g(n)$ explicitly for all $n \geq 1$ for $g = 0$ and $g = 1$. Same for small g and n.
- If $\Gamma_g(n)$ is connected, must $\Gamma_g(n+1)$ be connected?
- For a given g, what is the minimal N so that $\Gamma_g(n)$ is connected for all $n \geq N$ (if such an N exists)?
- Study other graph-theoretic properties of $\Gamma_g(n)$ (many can be found via google, and also via any graph theory book).

Geom/Top: Homework 2 (due Monday, 10/15/12)

1. Read Farb notes.
2. Read Hatcher, Section 2.1.
3. (Don't hand in): Check that chain homotopy is an equivalence relation on the set of chain maps.

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1. Some problems on contractibility:

   (a) The **Dunce Hat** is the quotient space obtained from the standard 2-simplex $[v_0v_1v_2]$ by identifying $[v_0v_1]$ with $[v_0v_2]$ with $[v_1v_2]$ (note that the ordering matters!). Prove that the Dunce Hat is contractible.

   (b) Prove that the retract of a contractible space is contractible.

   (c) Prove that S^∞ is contractible.

   (d) Prove that the torus minus one point deformation retracts to $S^1 \vee S^1$.

   (e) Prove that a space Y is contractible if and only if for every space X, any two continuous maps $f, g : X \to Y$ are homotopic. Prove the same theorem but replacing $f : X \to Y$ with $f : Y \to X$.

2. Let $f, g : S^n \to S^n, n > 0$ be continuous maps with the property that $f(x)$ and $g(x)$ are not antipodal for any x. Prove that f and g are homotopic.

3. Hatcher page 19, Problem 20.

4. Let X be a space. The **suspension** SX is defined as the quotient space

$$SX := \frac{X \times [0,1]}{(X \times \{0\}) \sqcup (X \times \{1\})}$$

   (a) Let X be a Δ-complex. Prove that SX is a Δ-complex.

   (b) Prove that $\tilde{H}_i(X) \approx \tilde{H}_{i+1}(SX)$ for all $i \geq 0$.

5. Think of S^1 as the unit circle in the complex plane. Let $f, g : S^1 \to S^1$ be the maps $f(z) = z$ and $g(z) = z^2$.

   (a) Prove that f is not homotopic to g.

   (b) Let $F(z,t) : S^1 \times [0,1] \to S^1$ be defined by $F(z,t) = z^{t+1}$. Why isn't F a homotopy from f to g?

   (c) Classify all continuous maps $f : S^1 \to S^1$ up to homotopy.

6. Let X be any topological space.

   (a) Prove that for any element $c \in H_1(X)$ (singular homology) there is a continuous map $f : S^1 \to X$ so that $c = f_*([S^1])$, where $[S^1]$ is a generator of $H_1(S^1)$.

(b) Let X be a Δ-complex (for simplicity). Prove that for any $c \in H_2(X)$ (use simplicial or singular homology, as you like), there is a closed surface S and a continuous map $f : S \to X$ so that $c = f_*([S])$, where $[S]$ denotes a generator of the cyclic (by previous homework) group $H_2(S)$.

The problem of representing homology classes by manifolds is deep and important. It was studied by Rene Thom in his Fields Medal work (see his paper in Comment. Math. Helv. You just did dimensions $i = 1, 2$, but the problem gets harder for $i > 2$. Amazingly, cycles of dimension $i < 8$ can be represented by manifolds, but in general this is not true in dimension $i \geq 8$. But, Thom proved that it is true in rational homology (to be discussed later) up to a rational multiple.

7. Let X be a Δ-complex and let G be any abelian group. We define the simplicial homology of X **with coefficients in** A as follows. Let $\{C_n(X), \partial_n\}$ denote the complex of simplicial chains on X. The **complex of** G**-valued simplicial chains** is defined to be the collection of abelian groups $C_n(X; G) := C_n(X) \otimes G$ with boundary homomorphisms

$$\partial_n \otimes \mathrm{Id} : C_n(X) \otimes G \to C_{n-1}(X) \otimes G$$

Thus any element of $C_n(X; G)$ can be written as a finite sum $\sum_\sigma g_\sigma \sigma$ where $g_\sigma \in G$ and the sum is taken over all n-simplices σ of X. It is easy to check that $(\partial \otimes \mathrm{Id})^2 = 0$, so that $\{C_n(X; G), \partial_n\}$ is a chain complex. One defines the simplicial homology of X **with coefficients in** G, denoted $H_i(X; G)$, as the homology of this chain complex.

(a) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Z}/2\mathbf{Z}$.

(b) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Z}/3\mathbf{Z}$.

(c) Compute the homology of the Klein bottle with coefficients in $G = \mathbf{Q}$.

Extra credit problems

1. You can still hand in any extra credit problems from Homework 1.

2. For any $n \geq 1$, give a Δ-complex structure on S^n, and compute its simplicial homology groups.

Geom/Top: Homework 2 (due Monday, 10/15/12)

1. Suppose we have a homomorphism of short exact sequences of abelian groups:

$$\begin{array}{ccccc} 0 \to & A \to & B \to & C \to & 0 \\ & \downarrow f & \downarrow g & \downarrow h & \\ 0 \to & A' \to & B' \to & C' \to & 0 \end{array}$$

   Prove that there is an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \mathrm{cok}(f) \to \mathrm{cok}(g) \to \mathrm{cok}(h) \to 0$$

2. Let X be any topological space, and let $A \subseteq X$ be a nonempty subspace.

   (a) Prove directly (i.e. not using any long exact sequences, but only the definitions), that if $x \in X$ is any point, then

$$H_i(X, x) \approx \tilde{H}_i(X) \text{ for all } i \geq 0$$

   (b) Suppose that there is a retraction $r : X \to A$. Show that

$$H_i(X) \approx H_i(A) \oplus H_i(X, A)$$

   for all $i \geq 0$.

   (c) Prove that if the inclusion $i : A \to X$ is a homotopy equivalence then then $H_i(X, A) = 0$ for all $i \geq 0$.

   (d) Prove that the inclusion $A \to X$ induces isomorphisms on all homology groups if and only if $H_i(X, A) = 0$ for all $i \geq 0$.

3. Give an example of pairs of spaces (X, X_0) and (Y, Y_0) where $H_i(X) \approx H_i(Y)$ and $H_i(X_0) \approx H_i(Y_0)$ but $H_i(X, X_0) \not\approx H_i(Y, Y_0)$.

4. Compute the following examples of relative homology groups of the following pairs (X, A) using the long exact sequence of a pair. Note how the answer compares to the homology of the quotient X/A.

   (a) An annulus relative to its boundary (union of 2 circles).

   (b) A Mobius band relative to its boundary. Give a generator for relative H_1 in this case.

   (c) (S^n, S^{n-1}) for each $n \geq 0$.

5. Prove for an n-manifold ($n \geq 1$), and any point $x \in X$, that

$$H_i(X, X - x) = \begin{cases} \mathbf{Z} & i = 0, n \\ 0 & i \neq 0, n \end{cases}$$

6. Let $\gamma \subset S^2$ be a subset homeomorphic to $[0,1]$, and let $x \in \gamma$ be any point. Prove that the inclusion of pairs $(S^2 - x, \gamma - x) \to (S^2, \gamma)$ is not an excision.

7. Check that the "connecting homomorphism" $\partial : H_n(X,A) \to H_n(A)$ in the long exact sequence of a pair (X,A), defined purely via homological algebra, has a geometric meaning: if $c \in Z_n(X,A)$ is a cycle representing an element of $H_n(X,A)$, then the element $\partial[c] \in H_{n-1}(A)$ is represented by the cycle $\partial c \in Z_{n-1}(A)$.

8. Assuming that G is abelian, what can you say about the isomorphism type of G if

   (a) $0 \to \mathbf{Z}^n \to G \to \mathbf{Z}^m \to 0$ is exact?

   (b) $0 \to \mathbf{Z}/4\mathbf{Z} \to G \to \mathbf{Z}/4\mathbf{Z} \to 0$ is exact?

9. Let X be any space, and let

$$0 \to G \to G' \to G^{''} \to 0$$

   be a short exact sequence of abelian groups.

   (a) Prove that there is a short exact sequence of chain complexes (with varying coefficients):

$$0 \to C_n(X;G) \to C_n(X;G') \to C_n(X;G^{''} \to 0$$

   By the Fundamental Theorem of Homological Algebra one gets the induced long exact sequence of homology groups. Denote the associated "connecting homomorphism" by

$$\beta : H_n(X;G^{''}) \to H_{n-1}(X;G)$$

   (b) Compute β when X is the Klein bottle and the coefficient sequence is

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

   (c) Do the same for the coefficients

$$0 \to \mathbf{Z}/2\mathbf{Z}/ \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

Extra Credit Problems

1. You can still hand in any extra credit problems from Homeworks 1 and 2.

Geom/Top: Homework 4 (due Monday, 10/29/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Let X be a connected space which is a finite union of polygons $\{P_i\}$ with the property that the intersection of any two polygons is either empty, a common edge, or a common vertex. Let V, E and F denote the total number of vertices, edges, and faces of these polygons (so e.g. a face is precisely the interior of some polygon).

   (a) Prove that $\chi(X) = V - E + F$.

   (b) The space X is called a **regular polytope** if all the P_i have the same number r of edges, if each edge in X lies in exactly two faces, and if each vertex in X lies in some fixed number s of faces. Prove that there are precisely five regular polytopes homeomorphic to S^2.

   (c) What are the possibilities for r and s when X is homeomorphic to the torus T^2?

2. Let X be any finite, connected Δ-complex. Prove that $f_* : H_0(X) \to H_0(X)$ is the identity homomorphism for any continuous map $f : X \to X$.

3. Let Σ_g denote the closed, connected, genus $g \geq 0$ surface.

   (a) For each $g \neq 1$, find a homeomorphism $f : \Sigma_g \to \Sigma_g$ with no fixed point.

   (b) Prove that for $g \neq 1$ any homeomorphism $f : \Sigma_g \to \Sigma_g$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

4. Let G be a path-connected, compact **topological group**. That is, G is a group and also a compact topological space, such that the maps $G \times G \to G$ with $(a, b) \mapsto ab$ and $G \to G$ with $g \to g^{-1}$ are continuous. Assume that G has some Δ-complex structure.

   (a) For any $g \in G$, let $L_g : G \to G$ be "left translation by g", i.e. $L_g(h) = gh$. Prove that L_g is homotopic to the identity.

   (b) Conclude that $\chi(G) = 0$.

   (c) Prove that $S^{2n}, n > 0$ cannot be given the structure of a topological group.

   (d) Prove that the only compact surface which is a topological group is the torus; in particular rule out the Klein bottle. [Warning: I am not 100% sure the Klein bottle part can be solved at this point.]

5. Give an example of a finite Δ-complex X and a continuous self-map $f : X \to X$ such that f has a fixed point but the Lefschetz number $\Lambda(f) = 0$. Thus the converse of the Lefschetz Fixed Point Theorem does not hold.

6. Vector fields $\{V_i\}$ on S^n are **linearly independent** if for each $z \in S^n$ the vectors $\{V_i(z)\}$ are linearly independent in the vector space TS^n_z.

   (a) Recall that in class we found a nonvanishing vector field on S^{2n+1}. Adapt this construction to give 3 linearly independent nonvanishing vector fields on S^{4n+3}.

   (b) Construct 7 linearly independent vector fields on S^7.

   (c) Generalize these constructions to produce the maximal number of possible linearly independent vector fields on S^n for each n, where the upper bound is given by Adams's Theorem.

7. Think of S^n as the set of unit vectors v in $\mathbf{R}^{n+1}$. Consider the question: when does there exist a continuous map $f : S^m \to S^n$ satisfying $f(-v) = -f(v)$, that is, preserving the property of points being antipodal to each other. Note that when $m < n$ this is trivial: just let $f : S^m \to S^n$ be a standard inclusion.

   Theorem (Borsuk-Ulam): *When $m > n$ there does not exist any continuous map $f : S^m \to S^n$ satisfying $f(-v) = -f(v)$ for all $v \in S^m$.*

   Assume for now this theorem.

   (a) Prove that any continuous map $f : S^n \to \mathbf{R}^n$ there exists $v \in S^n$ with $f(v) = f(-v)$. This statement is the more frequently stated "Borsuk-Ulam Theorem". Deduce that S^n does not embed in $\mathbf{R}^n$.

   (b) Use (a) to prove invariance of dimension: $\mathbf{R}^m \approx \mathbf{R}^n$ implies $m = n$.

   (c) Prove the following: Let $\{X_1, \ldots, X_{n+1}\}$ be a covering of $S^n, n > 0$ by closed sets. Then some X_i contains a pair of antipodal points. [Hint: Consider the map $f : S^n \to \mathbf{R}^n$ given by

   $$f(v) = (d(v, X_1), \ldots, d(v, X_n))$$

   where $d(v, X_i)$ denotes the closest distance in S^n from v to some any point in the set X_i.]

8. Recall the Fundamental Theorem of Algebra: Every polynomial $P(x) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_i \in \mathbf{C}$ has a zero in $\mathbf{C}$. Prove this theorem as follows: Let S_r denote the circle of radius r in $\mathbf{C}$. Suppose P has no zero inside S_r. Then we can think of the restriction $f = P|_{S_r}$ of P to S_r as a continuous map $f : S_r \to \mathbf{C} - 0$.

   (a) Prove that $f_* : H_1(S_r) \to H_1(\mathbf{C} - 0)$ is trivial.

   (b) Prove that for r sufficiently large, f is homotopic to to the map $z \mapsto z^n$. [Hint: Let $F_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$.]

   (c) Derive a contradiction.

Extra Credit Problems

1. Let X be any finite Δ-complex with $\chi(X) \neq 0$. Prove that any homeomorphism $f : X \to X$ has a periodic point, i.e. some power $f^n, n > 0$, has a fixed point.

2. Prove that if M is a closed manifold that admits a nonvanishing vector field, then $\chi(M) = 0$.

Geom/Top: Homework 5 (due Monday, 11/12/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Let $\{G_n\}$ be any sequence of finitely generated abelian groups. Prove that there is a CW complex X, with finite n-skeleton for each n, so that $H_n(X) = G_n$ for $n \geq 1$.

2. Find spaces $X = A \cup B$ and $X' = A' \cup B'$ so that for all $n \geq 0$:

   $$H_n(A) \approx H_n(A') \quad \text{and} \quad H_n(B) \approx H_n(B') \quad \text{and} \quad H_n(A \cap B) \approx H_n(A' \cap B')$$

   but with some $j > 0$ for which $H_j(X)$ not isomorphic to $H_j(X')$. Thus the "connecting homomorphism" in the Mayer-Vietoris sequence does matter when computing the homology of X.

3. Solve the bonus problem on the exam.

4. (a) Compute the homology groups of $\mathbf{R}^2$ minus k-points, for any $k \geq 1$.

   (b) Compute the homology of $\mathbf{R}^3$ minus k non-intersecting lines.

5. Let $U(2)$ denote the group of complex 2×2 unitary matrices. Endow $U(2)$ with the subspace topology $U(2) \subset \mathbf{C}^4$.

   (a) Let $SU(2)$ be the subgroup (and topological subspace) of $U(2)$ consisting of those matrices of determinant 1. Prove that $SU(2)$ is homeomorphic to S^3.

   (b) Consider the determinant $\det : U(2) \to S^1$ map. Let $A \subset U(2)$ (resp. B) be the subset of elements with determinant $e^{i\theta}, -\pi < \theta < \pi$ (resp. $0 < \theta < 2\pi$). Compute $H_n(U(2))$ for all $n \geq 2$ by applying Mayer-Vietoris to $X = A \cup B$.

6. Hatcher, §3.1, Problems 4,5,

Extra Credit Problems

1. Let $\mathrm{Conf_n}(\mathbf{C})$ be the set of (unordered) configurations of n distinct points in $\mathbf{C}$, topologized as a subset of $\mathbf{C}^n$.

   (a) Prove that $\mathrm{Conf_n}(\mathbf{C})$ is the space of monic, square-free, polynomials of degree n. [Worth 2 points.]

   (b) Prove that $\mathrm{Conf_n}(\mathbf{C})$ is the complement in $\mathbf{C}^n$ of a collection of hyperplanes. [Worth 2 points.]

   (c) Compute the homology of $\mathrm{Conf_n}(\mathbf{C})$. [Worth 30 points.]

Geom/Top: Homework 5 (due Monday, 11/19/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Hatcher, Section 3.2, Problems 3, 12, 15 (compute the Poincaré series of $S^n, \mathbf{RP}^n, \mathbf{RP}^\infty$ only).
2. Compute the cohomology ring of $\mathbf{RP}^2 \times \mathbf{RP}^2$.
3. Prove that the Klein bottle and the space $X = \mathbf{RP}^2 \vee S^1$ have isomorphic cohomomology rings with $\mathbf{Z}$ coefficients, but not with $\mathbf{Z}/2\mathbf{Z}$ coefficients.
4. Let $L_1 \subset S^3$ be the union of two disjoint round circles that link each other once. Let L_2 be the union of two disjoint , unlinked round circles . Prove that $S^3 - L_1$ and $S^3 - L_2$ has isomorphic integral cohomology groups, but not isomorphic cohomology rings.

Geom/Top: Homework 7 (due Monday, 11/26/12)

1. Read Farb notes.

2. Read along in Hatcher.

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1. Recall that a *bilinear pairing* of groups or vector spaces is a linear map $V \times W \to \mathbf{R}$ which is *nondegenerate* in the sense that for each nonzero $v \in V$, the map $v \mapsto < v, - >$ is an isomorphism $V \to W^*$.

   Give another proof of (an equivalent form of) Poincaré Duality by proving that cup product gives a nondegenerate pairing

   $$H^i(M) \times H_c^{n-i} \to H_c^n(M)$$

   so that $H_c^{n-i} \approx (H^i)^*$.

2. Let S_g denote the closed, orientable surface of genus g. Prove that if $g < h$ then any continuous map $f : S_g \to S_h$ has degree 0. [Hint: Measure degree via the action on $H^2(S_g; \mathbf{Z})$. Prove that there exists $a \in H^1(S_h; \mathbf{Z})$ with $f^*a = 0$. On the other show that there exists b with $a \cup b \neq 0$.]

3. Let M be a compact, connected, nonorientable 3-manifold. Prove that $H_1(M)$ is infinite.

4. Look up Poincaré Duality for manifolds with boundary. Prove that if M is the boundary of some compact connected manifold, then $\chi(M)$ is even. Give an example of a closed manifold that is not the boundary of any compact manifold.

Geom/Top: Homework 8 (due Monday, 12/03/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Hatcher, §1.1, Problems 5, 6, 7, 16(c)(f).
2. Let X be a path-connected space, and let $x \in x$. Let $\phi : \pi_1(X, x) \to H_1(X, \mathbf{Z})$ be the map $\phi([\gamma]) := \gamma_*(S^1)$, where we think of $\gamma : S^1 \to X$. Prove that ϕ is surjective, and that the kernel of ϕ is precisely the commutator subgroup of $\pi_1(X, x)$.
3. Let G be a path-connected topological group, and suppose that $\pi_1(G) = 0$. Let Γ be a discrete normal subgroup of G. Prove that $\pi_1(G/\Gamma) \approx \Gamma$. Note that the case $G = \mathbf{R}, \Gamma = \mathbf{Z}$ gives another proof that $\pi_1(S^1) \approx \mathbf{Z}$.
4. Now let G be any connected topological group, and let Γ be any discrete normal subgroup. Prove that Prove that Γ is central in G (i.e. each $g \in \Gamma$ commutes with every element of G). Deduce that $\pi_1(G/\Gamma)$ is abelian. In particular this proves that $\pi_1(G)$ is abelian.
5. Let $p : S^n \to \mathbf{RP}^n$ be the standard quotient map sending v to $\{\pm v\}$. Prove by hand, just as in the proof that $\pi_1(S^1) \approx \mathbf{Z}$, that $\pi_1(\mathbf{RP}^2) \approx \mathbf{Z}/2\mathbf{Z}$.

Geom/Top: Homework 9 (due Monday, 12/10/12)

1. Read Farb notes.
2. Read along in Hatcher.

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1. Hatcher, §1.2, Problems 4,10, 22.
2. Hatcher §1.3 Problems 4,9,10, 14.
3. Let S_g be a surface of genus $g \geq 0$. Let $\Delta \subset S_g \times S_g$ be the image of the diagonal embedding $x \mapsto (x, x)$. Let X be the complement of Δ in $S_g \times S_g$. Compute $\pi_1(X)$.