NOTES ON THE SURGERY OBSTRUCTION

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1. Introduction

The goal of this talk is to describe a small portion of the answer to the following question:

Question 1. When is a simply connected space X homotopy equivalent to a (compact) n-dimensional smooth manifold?

A compact manifold is homotopy equivalent to a finite CW complex, so X must be one itself. Equivalently (since X is simply connected), the homology $H_*(X;\mathbb{Z})$ must be finitely generated.

More interestingly, we know that a hypothetical compact n-manifold M homotopy equivalent to X (which is necessarily orientable) satisfies *Poincaré duality.* That is, we have a *fundamental class* $[M] \in H_n(M;\mathbb{Z})$ with the property that cap product induces an isomorphism

$$H^r(M;G) \overset{\cap[M]}{\simeq} H_{n-r}(M;G),$$

for all groups G and for all r. It follows that, if the answer to the above question is positive, an analogous condition must hold for X. This motivates the following definition:

Definition 1. A simply connected finite CW complex X is a **Poincaré duality space (or Poincaré complex) of dimension** n if there is a class $[X] \in H_n(X;\mathbb{Z})$ such that for every abelian group G, cap product induces an isomorphism

$$H^r(X;G) \overset{\cap[M]}{\simeq} H_{n-r}(X;G).$$

A consequence is that the cohomology (and homology) groups of X vanish above dimension n, and $H_n(X;\mathbb{Z})$ is generated by $[X]$. In other words, $[X]$ behaves like the fundamental class of an n-dimensional manifold.

We can thus pose a refined version of the above question.

Question 2. Given a simply connected Poincaré duality space X of dimension n, is there a homotopy equivalence $f : M \to X$ for M a compact (necessarily n-dimensional) manifold?

The answer to Question 2 is not always positive.

Example 1. The Kervaire manifold ([Ker60]) is a topological 4-connected 10-manifold of (suitably defined) Kervaire invariant one. Since, as Kervaire showed in that paper, all smooth framed 10-manfolds have Kervaire invariant zero, he concluded that there was no smooth manifold in its homotopy type.

Example 2. It is known that there exists a simply connected compact *topological* 4-manifold whose intersection form is even and has signature eight (the "E_8 manifold"). Such a manifold cannot be homotopy equivalent to any smooth manifold. In fact, if it were homotopy equivalent to a smooth 4-manifold M, then the evenness of the intersection form on $H^2(M;\mathbb{Z})$ shows that Sq^2 (and in fact all $\mathrm{Sq}^i, i \neq 0$) act trivially on M. The Wu formulas imply that the Stiefel-Whitney classes of M vanish. In particular, M admits a *spin structure*, and a theorem of Rochlin (see [MK60]) asserts that the signature of a spin 4-manifold is divisible by 16.

Example 3 (See also [MM79])**.** An example (due to Gitler and Stasheff) of a four-cell Poincaré complex which is not homotopy equivalent to a manifold is given by

$$X = (S^2 \vee S^3) \cup_f e^5,$$

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where $f : S^4 \to S^2 \vee S^3$ is the sum of the Whitehead product $[\iota_2, \iota_3]$ (in other words, the attaching map for the top cell in $S^2 \times S^3$) and the map $S^4 \xrightarrow{\eta^2} S^2 \to S^2 \vee S^3$.

One sees that the cup product of the class in dimension two and the class in dimension three is the one in dimension five, and consequently X is a Poincaré complex. In fact, this is a consequence of the fact that X maps to $(S^2 \vee S^3) \cup_{[\iota_2,\iota_3],\eta^2} (e^5 \vee e^5)$, and the cohomology ring of this space follows from the structure of $S^2 \times S^3$ and $e^5 \cup_{\eta^2} S^2$.

Observe now that, stably,

$$\Sigma^\infty_+ X \simeq (e^5 \cup_{\eta^2} S^2) \vee S^3 \vee S^0,$$

because the suspension of $[\iota_2, \iota_3]$ vanishes: $S^2 \times S^3$, or any product, splits stably. In particular, the Steenrod squares in $H^*(X; \mathbb{Z}/2)$ all vanish.

The claim is that X is not homotopy equivalent to a manifold. In fact, if it were, the fact that Steenrod squares are zero in $H^*(X; \mathbb{Z}/2)$ implies that the Stiefel-Whitney classes of X vanish by the Wu formula: thus the stable normal bundle of X acquires a *spin structure*. Atiyah duality (see the next section) would show that the Spanier-Whitehead dual of $\Sigma^\infty_+ X$, which is (up to suspension)

$$(e^3 \cup_{\eta^2} S^0) \vee S^2 \vee S^5,$$

would be the Thom spectrum of the (spin) stable normal bundle. (Note that the suspension of $[\iota_2, \iota_3]$ vanishes: $S^2 \times S^3$, or any product, splits stably.) As the Thom spectrum of a spin bundle, it would come with a map

$$(e^3 \cup_{\eta^2} S^0) \vee S^2 \vee S^5 \to M\mathrm{Spin}$$

which is an isomorphism on π_0. In particular, we would get a map

$$(e^3 \cup_{\eta^2} S^0) \to M\mathrm{Spin},$$

which hits the bottom cell of MSpin.

Note that the cohomology groups $H^i(\mathrm{MSpin}; \mathbb{Z}/2)$ vanish for $i = 1, 2, 3$, because Spin is 2-connected and consequently BSpin is 3-connected. The Adem operation $\Phi : H^0 \to H^{i+3}$ (a *secondary cohomology operation*) acts trivially on the bottom cohomology class of MSpin because of this connectivity but is nontrivial in $(e^3 \cup_{\eta^2} S^0)$. This is a contradiction.

2. Atiyah duality and Spivak fibrations

Let us return to the question above. Consider a simply connected Poincaré duality space X of dimension n. If X has the homotopy type of a manifold, we should in particular obtain an n-dimensional vector bundle ξ on X, coming from the tangent bundle of that manifold. We cannot expect this bundle to be uniquely determined, because the tangent bundle of a smooth manifold is not a homotopy invariant. Nonetheless, ξ is not totally arbitrary either. It turns out the *stable spherical fibration* associated to ξ is homotopy invariant, and can in fact be reconstructed simply from X.

To see this, let's recall the homotopy properties of the tangent bundle. Let M be a smooth, compact manifold. The tangent bundle TM cannot be reconstructed homotopy-theoretically from M. However, one important homotopy-theoretic property of the *normal* bundle ν of an imbedding $M \hookrightarrow S^N$ is the *Pontryagin-Thom collapse map*

$$S^N \to M^\nu,$$

mapping onto the *Thom complex* M^ν of ν. It crushes everything outside a tubular neighborhood of M to the "point at infinity" in the Thom complex. It has the property that $\widetilde{H}_N(S^N, \mathbb{Z}) \to \widetilde{H}_N(M^\nu, \mathbb{Z})$ has the property that $[S^N]$ is sent to the generator of $\widetilde{H}_N(M^\nu; \mathbb{Z})$ (corresponding to the fundamental class of M). This is called "S-reducibility" and can be phrased by saying that the top cell of M^ν splits off.

We can use the Pontryagin-Thom map to produce a map

$$S^N \to M^\nu \to M^\nu \wedge M_+$$

and desuspending produces a map of *spectra*

$$\mathrm{coev} : S^0 \to M^{-TM} \wedge \Sigma^\infty_+ M,$$

where M^{-TM} is the *Thom spectrum* $M^{-TM} \overset{\text{def}}{=} \Sigma^{-N} M^{\nu}$. There is also a map in the opposite direction,

$$M^{\nu} \wedge M_+ \to S^N,$$

obtained as follows. First, $M^{\nu} \wedge M_+ \simeq (M \times M)^{p_1^* \nu}$. There is a (diagonal) submanifold $M \overset{\Delta}{\to} M \times M$ whose normal bundle is isomorphic to the tangent bundle of M. That gives a Pontryagin-Thom collapse map

$$M \times M \to M^{TM}$$

crushing the exterior of a tubular neighborhood of the diagonal, and therefore a map

$$M^{\nu} \wedge M_+ \simeq (M \times M)^{p_1^* \nu} \to M^{\nu \oplus TM} \to S^N$$

where the final map is $M^{\nu \oplus TM} \simeq M^{\mathbb{R}^n} \to S^n$ that crushes M to a point. Desuspending produces a map of Thom spectra

$$\mathrm{ev} : M^{-TM} \wedge \Sigma^{\infty}_+ M \to S^0.$$

Theorem 1 (Atiyah [Ati61])**.** *Let M be a compact smooth manifold. The Spanier-Whitehead dual to $\Sigma^{\infty}_+ M$ is the Thom spectrum M^{-TM}, under the above maps* $\mathrm{coev} : S^0 \to M^{-TM} \wedge \Sigma^{\infty}_+ M$ *and* $\mathrm{ev} : M^{-TM} \wedge \Sigma^{\infty}_+ M \to S^0$.

This result can be viewed as a refinement of Poincaré duality: it implies, for instance, the Poincaré duality theorem in generalized cohomology. It is, by definition, the claim that the composites

$$S^N \wedge M^{\nu} \overset{\Sigma^N \mathrm{coev} \wedge 1}{\longrightarrow} (M^{\nu} \wedge M_+) \wedge M^{\nu} \simeq M^{\nu} \wedge (M_+ \wedge M^{\nu}) \overset{1 \wedge \Sigma^N \mathrm{ev}}{\longrightarrow} M^{\nu} \wedge S^N \simeq S^N \wedge M^{\nu}$$

and

$$S^N \wedge M_+ \overset{\Sigma^N \mathrm{coev} \wedge 1}{\longrightarrow} (M_+ \wedge M^{\nu}) \wedge M_+ \simeq M_+ \wedge (M^{\nu} \wedge M_+) \overset{1 \wedge \Sigma^N \mathrm{ev}}{\longrightarrow} M_+ \wedge S^N \simeq S^N \wedge M_+$$

are homotopic to the identity (at least stably). One can verify this by looking at preimages of the zero section: a map of Thom spaces is determined at what it does near the zero section.

The Thom spectrum M^{-TM} is determined by M solely in terms of (stable) homotopy theory. However, the Thom spectrum associated to a vector bundle E over a space X does not determine the vector bundle. For one thing, it depends only on the class of the vector bundle in KO-theory. It even depends only on the stable spherical fibration associated to the vector bundle, because the Thom complex can be described (up to homotopy) as the mapping cone of

$$\mathrm{Sph}(E) \to X.$$

The next proposition shows that, on a manifold, this is the only indeterminacy: that is, the stable spherical fibration associated to the normal (or tangent) bundle is determined by homotopy theory.

Proposition 2. *Let M be a manifold, and let E be a vector bundle on M such that M^E is S-reducible. Then the stable spherical fibration associated to E is homotopy equivalent to the one associated to the stable normal bundle. In particular, the stable spherical fibration associated to TM is a homotopy invariant of M.*

Proof. The Spanier-Whitehead dual of $\Sigma^{\infty} M^E$ is M^{-TE-E}: one can produce the appropriate maps as above by taking Thom spaces. By hypothesis, there is a map of spectra

$$S^N \to \Sigma^{\infty} M^E, \quad N = \dim M + \dim E$$

splitting off the top cell. Taking duals produces a stable map

$$M^{-TE-E} \to S^{-N}.$$

We can represent this by an actual map of spaces

$$M^{n-TE-E} \to S^{n-N}, \quad n \gg N,$$

where $n - TE - E$ is an honest (not virtual) vector bundle for n large enough. By assumption, this map splits off the bottom cell.

Now, let V be the vector bundle $n - TE - E$: for $n \gg 0$ it is actually well-defined. Let $B(V)$ be the ball bundle of V, and $S(V)$ be the sphere bundle; we have $M^V = B(V)/S(V)$. By assumption, there is a map of pairs

$$(B(V), S(V)) \to (B^{\dim V}/S^{\dim V-1}, *)$$

such that, when restricted to the fiber $(B(V_x), S(V_x))$ over any $x \in M$, it induces a degree one map $B(V_x)/S(V_x) \to S^{\dim V}$. When $\dim V$ is large, we can replace this by a map of pairs

$$(B(V), S(V)) \to (B^{\dim V}, S^{\dim V-1}).$$

To see this, observe that the map $M \simeq B(V) \to S^{\dim V}$ is nullhomotopic (for $\dim V \gg 0$), so we can extend this to a map $\Sigma S(V) \to S^{\dim V}$. If $\dim V \gg 0$, the Freudenthal suspension theorem lets us desuspend this to a map $S(V) \to S^{\dim V-1}$.

We in particular get a map

$$S(V) \to S^{\dim V-1}$$

which restricts to an equivalence over any $x \in M$. But this is precisely a trivialization of the spherical fibration $S(V) \to M$. Since V is a trivial bundle plus $-TE - E$, we find that E is stably fiber homotopy equivalent to the normal bundle of M. □

The definition suggests that there might be an analog of the spherical fibration to a general Poincaré duality, and that turns out to be the case. For our purposes, we will take the S-reducibility of the Thom space of the normal bundle as a distinguishing feature.

Definition 2. Let X be a simply connected Poincaré complex. A **Spivak fibration** for X is a spherical fibration $T \to X$ such that the Thom complex X^T of T (that is, the mapping cone of $T \to X$) admits a map

$$S^N \to X^T$$

sending $[S^N] \in H_N(S^N; \mathbb{Z})$ to the image of the fundamental class $[X]$ under the Thom isomorphism.[1]

It can be shown that any Poincaré complex admits a Spivak fibration, and that it is (stably) unique. For our purposes, we will take this mostly as motivation.

3. Statement of the main result

Let us suppose now that there exists a vector bundle ξ over the simply connected Poincaré complex X, with the property that there exists a stable map

$$S^N \to X^\xi$$

sending $[S^N]$ to the fundamental class $[X]$ (fed into the Thom isomorphism): in other words, ξ is a vector bundle lifting the Spivak normal fibration. Is this enough to show that X is homotopy equivalent to a manifold M such that ξ corresponds to the stable normal bundle? Suppose $\dim X = 4k$. Then there is an obstruction that comes from cobordism theory. One has:

Theorem 3 (Hirzebruch signature formula)**.** *If M is an oriented $4k$-dimensional manifold, then there exists a polynomial $\mathbf{L}(x_1, \dots, x_k) \in \mathbb{Q}[x_1, \dots, x_k]$ such that*

$$\sigma(M) = \int_M \mathbf{L}(p_1, \dots, p_k).$$

In other words, the signature $\sigma(M)$ of M can be computed in terms of the Pontryagin classes p_i of the tangent bundle of M.

In the above situation, the signature is a homotopy invariant: in particular, we should expect the signature formula to hold for (X, ξ) as well if we are to realize (X, ξ) from a manifold. The following result states that it is the *only* obstruction.

Theorem 4 (Browder-Novikov [Bro95])**.** *Let $k \geq 2$. Then a simply connected, $4k$-dimensional Poincaré complex X is homotopy equivalent to a manifold if and only if there exists a stable bundle ξ on M such that:*

[1] Since X is simply connected, we have a fibration of oriented spheres, and there is a Thom isomorphism.

(1) The complex X^ξ is S-reducible: there exists a map $S^N \to X^\xi$ inducing an isomorphism on top-dimensional homology.
(2) Hirzebruch's signature formula is valid for the pair (X, ξ).

The goal of this talk is to sketch the proof of this result. The proof proceeds in two stages. The first stage uses cobordism theory to produce a *degree one normal map*

$$f : M \to X$$

from a $4k$-dimensional manifold: in other words, $f_*([M]) = [X]$ and there is an isomorphism between $f^*\xi$ and the stable normal bundle of M. The second (more involved) stage involves doing *surgery* on M (actually, on f) to make the map a homotopy equivalence, while preserving these conditions. The condition on the signature is necessary to make the last step of the surgery work.

4. Degree one normal maps

Let X be a Poincaré complex satisfying the first condition of the statement of the Browder-Novikov theorem. Our goal is to produce a degree one normal map $f : M \to X$. In order to do this, we can use the Pontryagin-Thom construction. Namely, recall that we have a map

$$\phi : S^N \to X^\xi$$

inducing an isomorphism on top homology. Here $X \hookrightarrow X^\xi$ via the zero section, and a small neighborhood of X in X^ξ.

Using a transversality argument, we can make ϕ transverse to the zero section. In this case, $M \stackrel{\text{def}}{=} \phi^{-1}(X)$ is a manifold, whose normal bundle (in S^N) is isomorphic to the pull-back ξ, under the map $f : M \to X$ that we get. Moreover, we have a map of pairs

$$(S^N, S^N \setminus M) \to (X^\xi, X^\xi \setminus X)$$

and a look at this map, and the naturality of the Thom isomorphism, shows that the map

$$f : M \to X$$

is a *degree one normal map.* That is, $f_*([M]) = [X]$); that follows by looking at the image of $[S^N]$ in $H_*(X^\xi; \mathbb{Z})$.

We find:

Proposition 5. *If X is a Poincaré complex, a lift ξ of the Spivak normal fibration (equivalently, a bundle ξ such that M^ξ is S-reducible) produces a degree one normal map $f : M \to X$ from a manifold.*

The Spivak normal fibration is a stable spherical fibration, and is therefore classified by a map

$$X \to \varinjlim B\mathrm{Haut}(S^n, S^n),$$

to the colimit of the classifying spaces of the homotopy self-equivalences of S^n. A choice of vector bundle lifting the Spivak fibration is a lift of the map $X \to \varinjlim B\mathrm{Haut}(S^n, S^n)$ under the *J-homomorphism* $BSO \to \varinjlim B\mathrm{Haut}(S^n, S^n)$.

5. Surgery below the middle dimension

Let X be a Poincaré complex. Given a lift of the Spivak normal fibration to a vector bundle on X, we saw in the previous section that we get a degree one normal map $f : M \to X$. Our goal is to modify f so as to make it closer to a homotopy equivalence; in order to do so we need a method of modifying the homotopy type of a manifold. This method is *surgery.*

Definition 3. Observe that

$$\partial(S^p \times D^{q+1}) = \partial(D^{p+1} \times S^q) = S^p \times S^q.$$

A ***p*-surgery** on the (oriented) n-manifold M consists of taking an (oriented) imbedding $S^p \times D^{q+1}$ (where $p+q+1 = n$), cutting out the image of the interior, and pasting in a copy of $D^{p+1} \times S^q$ along the common boundary. We get a new manifold $M' = (M - S^p \times \mathrm{Int} D^{q+1}) \cup_{S^p \times S^q} D^{p+1} \times S^q$.

Given any imbedding $S^p \times D^{q+1} \hookrightarrow M$, we can perform a p-surgery. We can also go in the other direction: after a p-surgery on M, we can always perform a q-surgery to get back to M (where $p + q + 1 = \dim M$ as before).

The next definition shows that the result is always oriented cobordant to M.

Definition 4. The **trace** of a p-surgery as above consists of $M \times I \cup_{S^p \times D^{q+1}} D^{p+1} \times D^{q+1}$. This is a manifold-with-boundary (obtained from $M \times I$ by "attaching a handle") whose boundary is the disjoint union of M and the new manifold M'. Note that, up to homotopy, $W \simeq M \cup_{S^p} D^{p+1}$.

In fact, any two manifolds are related by a surgery if and only if they are oriented cobordant. This is a consequence of Morse theory ([Mil61]).

Example 4. Let M, M' be two n-dimensional manifolds. We can perform a zero-surgery on $M \sqcup M'$ (based on any imbedding $S^0 \to M \sqcup M'$ which takes one point to M and another to M'). The result is the *connected sum* $M \# M'$.

Example 5. The sphere S^2 has the property that the complement of a band of the equator is diffeomorphic to $S^0 \times D^2$. A 0-surgery replaces this with a $D^1 \times S^1$ and results in a torus. More generally, a 0-surgery on a compact orientable surface increases the genus by one.

Our goal is now to use surgery to start with the map $f : M \to X$ and replace it by a highly connected map. The degree of connectivity of the map can be measured by the homotopy groups $\pi_i(X, M)$: recall that these consist of homotopy classes of diagrams

$$\begin{array}{ccc} S^{i-1} & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow f \\ D^i & \longrightarrow & X \end{array} .$$

Given such a diagram, we have a class $\alpha \in \pi_{i-1}(M)$; this is the image under the boundary map. We would like to do surgery on α in such a way to get a manifold M' mapping to X (in such a way that it is still a normal degree one map) but so that $\pi_*(X, M')$ is smaller than $\pi_*(X, M)$, at least in the lowest dimension. This can be done up to the middle dimension, when an obstruction arises. For one thing, it will not always be possible to do the surgery, and even when it is, we have to take care that it does not complicate rather than simplify our manifold.

We will prove:

Proposition 6. *Consider a degree one normal map $f : M \to X$ between a $2m$-dimensional manifold M and a $2m$-dimensional Poincaré complex X with bundle ξ. Then, by surgery on f, we can produce a degree one normal map $\widetilde{f} : \widetilde{M} \to X$ such that $\pi_i(X, \widetilde{M}) = \pi_i(\widetilde{f}) = 0$ for $i \leq m$.*

Proof. As above, the strategy is to perform surgery on f. Suppose $i \leq m$ is minimal such that $\pi_i(X, M) \neq 0$ (it's possible $i = 0$), and choose a class $\alpha \in \pi_i(X, M)$, represented by a diagram as above. We will construct a degree one normal map $f' : M' \to X$ where M' is obtained by an $(i-1)$-surgery on M so as to kill α.

In order to arrange that M' also maps to X, we will extend the normal map $M \to X$ to a normal map $W \to X$, where W is the trace of the surgery on M. The benefit is that W is obtained by attaching a cell from $M \times I$, so one knows when and how to extend maps from $M \times I$ to W: it is in particular a homotopy problem. The following definition will be useful:

Definition 5. A **bordism** between degree one normal maps $f : M \to X, f' : M' \to X$, from n-dimensional manifolds M, M' to an n-dimensional Poincaré complex X with a lift ξ of the Spivak fibration, is a normal map $F : (W, M, M') \to (X \times I, X \times \{0\}, X \times \{1\})$, where W is a manifold-with-boundary with $\partial W = M \sqcup (-M')$. In particular,

(1) F is required to extend f and f'.
(2) An identification of the stable normal bundle of W with $F^*\xi$ is given, and it extends the identifications of $f^*\xi, f'^*\xi$.
(3) F carries the relative fundamental class in $H_{n+1}(W, M \sqcup M'; \mathbb{Z})$ into that of $H_{n+1}(X \times I, X \times \partial I)$.

The strategy is to start with M, and then to construct a *bordism* between the degree one normal map $f : M \to X$ and a better map $f' : M' \to X$ (which will be obtained via surgery).

In order to do this, we will start by observing that $\partial\alpha : S^{i-1} \to M$ can be represented by an imbedding, in view of Whitney's imbedding theorem. By abuse of notation, we will identify α with $\partial\alpha$, although α consists not only of $\partial\alpha$ but also of a trivialization of $f \circ \partial\alpha$.

We are given a choice of trivialization of $f \circ \alpha$, which determines a choice of trivialization of the composite

$$S^{i-1} \stackrel{\alpha}{\to} M \stackrel{f}{\to} X \stackrel{\xi}{\to} BO,$$

which classifies the stable normal bundle of α in M. We would like to choose a choice of trivialization of the *unstable* normal bundle of α in M compatible with this.

Observe that since the *stable* normal bundle of α is trivial, so is the *unstable* one; in fact, the map

$$\pi_{i-2}(SO(2m-i+1)) \to \pi_{i-2}(SO)$$

is an isomorphism. Moreover, we can choose a trivialization of the normal bundle of α which agrees stably with the trivialization of $(f\circ\alpha)^*\xi$. We can do this because trivializations of the unstable normal bundle of α are a torsor over $\pi_{i-1}(SO(2m-i+1))$ while those of the stable bundle are a torsor over $\pi_{i-1}(SO)$, and

$$\pi_{i-1}(SO(2m-i+1)) \to \pi_{i-1}(SO)$$

is an isomorphism, as $i \leq m$.

This choice of trivialization of the *unstable* normal bundle of α determines an imbedding

$$S^{i-1} \times D^{q+1} \hookrightarrow M, \quad q = \dim M - i.$$

by the tubular neighborhood theorem. We then proceed to do surgery on this imbedding.

The resulting trace of the surgery is the manifold-with-boundary

$$W = M \times I \cup_{S^{i-1}\times D^{q+1}} D^i \times D^{q+1}.$$

We would like to claim two things about W:

(1) The map $M \times I \to X \times I$ extends (canonically) to a map $F : W \to X \times I$ (carrying the boundary into $X \times \{0, 1\}$).
(2) The identification of the stable normal bundle of M with $f^*\xi$ extends to an identification of that of W with $F^*\xi$.

In order to see (1), observe that W has the homotopy type of an i-cell attached along α. To extend $M \to X$ along W is to give a null-homotopy of $f \circ \alpha$, which is precisely what we have been given.

In order to see (2), observe that the stable normal bundle on W is specified by the bundle $\nu \simeq f^*\xi$ on M and by a trivialization of $\nu|_{S^{i-1}}$: namely, by the trivialization given by the tubular neighborhood $S^{i-1} \times D^{q+1}$ of $\alpha(S^{i-1})$. But this trivialization coincides with the trivialization of $f^*\xi$ arising from the nullhomotopy of $f \circ \alpha$, by *construction* of the tubular neighborhood. Since these two trivializations coincide, we find that $M' \to X$ is still a normal map: even better, $W \to X \times I$ is a normal map. Moreover, the fundamental class in $(W, \partial W)$ maps to that of $(X \times I, X \times \partial I)$. This implies that the fundamental class of M' goes to that of X.

What can we say about the map $f' : M' \to X$ that we have produced? Observe that W is obtained by attaching an i-cell along α and $W \to X$ is based on the nullhomotopy given by α itself. It follows that α is killed in $\pi_i(X, W)$. More precisely, the map

$$\pi_k(X, M) \to \pi_k(X, W)$$

is an isomorphism for $k < i$, and a surjection for $k = i$ with α in the kernel. Moreover, up to homotopy, W is obtained from M' by attaching a $(q+1)$-cell, where $q = \dim M - i \geq i$. It follows that $\pi_k(X, M') \to \pi_k(X, W)$ is an isomorphism for $k \leq i$. Putting these together, we find that

$$\pi_k(X, M) \simeq \pi_k(X, M'), \quad k < i,$$

while $\pi_i(X, M')$ is smaller than $\pi_k(X, M)$. Note moreover that $M' \to X$ preserves fundamental classes: we have in fact not modified the top homology group by passing to M'.

Continuing this process, we can kill all the homotopy groups below the "middle dimension." ∎

Example 6. Let's take $X = S^{2m}$. To give a degree one normal map $f : M \to X$ is equivalent to giving a *framed* $2m$-manifold (as the map f comes from free by crushing the $(2m-1)$-skeleton). The claim is that one can perform *framed* surgery on M so as to produce a $(m-1)$-connected manifold. In other words, every framed $2m$-manifold is framed cobordant to a framed $(m-1)$-connected manifold (see [Mil61], where these ideas were first introduced). If we could continue framed surgery and get an m-connected manifold, Poincaré duality would imply that we had a homotopy sphere. It turns out that, in all but finitely many dimensions, the obstruction to doing so vanishes, and every framed cobordism class is represented by a homotopy sphere. We will see that this happens for m even in the next section. (For m odd, the question is equivalent to the *Kervaire invariant problem.*)

6. The surgery obstruction

It follows that, given an n-dimensional (for $n = 2m$) simply connected Poincaré complex X with a lift of the Spivak fibration to a vector bundle, we can produce a degree one normal map $f : M \to X$ which is m-connected. We now would like to find the obstruction to continuing this process. Note, however, that there is exactly *one* more step to getting a homotopy equivalence. This will follow from the next lemma and the following discussion.

Lemma 7. *Let $f : M \to X$ be a degree one map of Poincaré complexes. Then for any abelian group G,*

$$f_* : H_*(M; G) \to H_*(X; G)$$

is a split surjection.

Proof. In fact, we observe that there is a commutative diagram

$$\begin{array}{ccc} H_i(M;G) & \xrightarrow{f_*} & H_i(X;G) \\ \downarrow \simeq & & \downarrow \simeq \\ H^{n-i}(M;G) & \xleftarrow{f^*} & H^{n-i}(X;G) \end{array}$$

The splitting comes from chasing around the diagram, since the vertical maps are isomorphisms. □

Now suppose $f : M \to X$ is m-connected, for $m = \frac{n}{2}$. Then the Hurewicz theorem implies that $H_i(X; M; \mathbb{Z}) = 0$ for $i \leq m$ and that the Hurewicz map

$$\pi_{m+1}(X, M) \to H_{m+1}(X, M; \mathbb{Z})$$

is an isomorphism. In particular, we find that the split surjection $H_i(M; \mathbb{Z}) \to H_i(X; \mathbb{Z})$ is an isomorphism for $i < m$. If it were an isomorphism for $i = m$, then Poincaré duality would imply that we had a homotopy equivalence already.

So let $K = K(f) = \ker(H_m(M; \mathbb{Z}) \to H_m(X; \mathbb{Z})) \simeq \pi_{m+1}(X, M)$. The final step in the surgery process is to kill the group K. The theorem thus follows from:

Proposition 8. *Surgery on $f : M \to X$ can kill K if and only if X satisfies Hirzebruch's signature formula.*

Proof. The necessity of X's satisfying the signature formula, in order to be homotopy equivalent to a manifold, is evident. Conversely, suppose that X satisfies the signature formula. Then

$$\sigma(M) = \sigma(X)$$

since $M \to X$ is a normal map. We have a commutative diagram:

$$\begin{array}{cccccc} K & \longrightarrow & H_m(M;\mathbb{Z}) & \xrightarrow{f_*} & H_m(X;\mathbb{Z}) & , \\ & & \phi_M \downarrow \simeq & & \phi_X \downarrow \simeq & \\ & & H^m(M;\mathbb{Z}) & \xleftarrow{f^*} & H^m(X;\mathbb{Z}) & \end{array}$$

which identifies the group $H^m(M; \mathbb{Z})$ as the direct sum $K \oplus H^m(X; \mathbb{Z})$.

The claim is that $K \perp H^m(X;\mathbb{Z})$ under the bilinear form on $H^m(M;\mathbb{Z})$ given by the cup product. In fact, the form sends a pair $(a,b) \in H^m(M;\mathbb{Z})$ to the evaluation pairing

$$\left\langle \phi_M^{-1}(a), b \right\rangle, \quad \phi_M^{-1} : H^m(M;\mathbb{Z}) \simeq H_m(M;\mathbb{Z}).$$

If $b = f^*b'$ comes from $b' \in H^m(X;\mathbb{Z})$, then that's equivalent to evaluating

$$\left\langle \phi_M^{-1}(a), f^*b \right\rangle = \left\langle f_*\phi_M^{-1}(a), b \right\rangle$$

and consequently the elements of the form f^*b' are orthogonal to the image of $\phi_M(K) \subset H^m(X;\mathbb{Z})$.

It follows that the cup product defines a nondegenerate symmetric bilinear form on K itself. Its signature is zero, since $\sigma(M) = \sigma(X)$: we have decomposed $H^m(X;\mathbb{Z}) = H^m(M;\mathbb{Z}) \oplus K$ as *groups with bilinear forms*. The strategy is to use results from the theory of quadratic forms to show that K can be killed by surgery.

Lemma 9. *K is a free abelian group.*

Proof. For any abelian group G, we have $H_i(X,M;G) = 0$ for $i < m+1$, so $H_i(M;G) \to H_i(X;G)$ is an isomorphism for $i < m$. This implies that $H^{n-i}(X;G) \to H^{n-i}(M;G)$ is an isomorphism for such i, or $H^r(X;G) \to H^r(M;G)$ is an isomorphism for $r > m$. For example, taking $r = m+1, m+2$ we find that $H^{m+2}(X,M;G) = 0$. Using the universal coefficient theorem, it follows that there can be no torsion in K, or there would be an Ext term in $H^{m+2}(X,M;G) = 0$. □

It follows that K is a *free* abelian group, on which there is a nonsingular bilinear pairing with signature zero. The rest of the proof proceeds by using facts about quadratic forms to write K in a particularly convenient form. Then, a careful bookkeeping of the surgery process shows that we can kill (spherical representatives of the) the generators of K, one by one.

Let's recall the following definition.

Definition 6. A **lattice** is a free, finitely generated abelian group Q together with a symmetric bilinear form $Q \times Q \to \mathbb{Z}$ which is *unimodular* (i.e., establishes an isomorphism $Q \simeq \hom(Q,\mathbb{Z})$).

For example, the free part of the middle cohomology group of an oriented manifold of dimension divisible by four is a lattice. Moreover, K is a lattice.

Given an element $x \in K$, we know that we can represent x by a map $\alpha : S^m \to M$ whose composite $S^m \to M \to X$ is nullhomotopic. Using a strong form of the Whitney imbedding theorem, we can assume that α is an imbedding—here we need $m \geq 3$. In particular, the quadratic form evaluated on x, that is (x,x), is the self-intersection of S^m in M.

Lemma 10. *Suppose m is even. Then $(x,x) \in 2\mathbb{Z}$ for $x \in K$. That is, any imbedding $\alpha : S^m \hookrightarrow M$ whose composite $S^m \to M \to X$ is trivial has even self-intersection. Moreover, $(x,x) = 0$ if and only if the normal bundle of α is trivial.*

Proof. In fact, the self-intersection of S^m with itself inside M is the Euler class of the normal bundle to S^m inside M. We know that the normal bundle, which is classified by an element $t \in \pi_{m-1}(SO(m))$, is *stably* trivial because $S^m \stackrel{\alpha}{\to} M \to X$ is nullhomotopic and because $M \to X$ is a normal map. It follows that t is killed under the map

$$\pi_{m-1}(SO(m)) \to \pi_{m-1}(SO(m+1)) \simeq \pi_{m-1}(SO).$$

Via the fibration $SO(m) \to SO(m+1) \to S^m$, we find that t is in the image of the boundary map

$$\pi_m(S^m) \to \pi_{m-1}(SO(m)),$$

which sends $1 \in \pi_m(S^m)$ to the class of the tangent bundle of S^m. Consequently, the Euler class (x,x) of t (the normal bundle of $S^m \stackrel{\alpha}{\hookrightarrow} M$) is a multiple of the Euler characteristic of S^m, so $(x,x) \in 2\mathbb{Z}$. This multiple is trivial if and only if $(x,x) = 0$. □

The previous lemma establishes the connection between the algebra of the bilinear pairing on K and the geometry of the maps $\alpha : S^m \to M$. We now use the following fact about quadratic forms:

Lemma 11. *Let K be a lattice of dimension d. Suppose $b(x,x) \in 2\mathbb{Z}$ for $x \in K$, and further that the signature of b (on $K \otimes_{\mathbb{Z}} \mathbb{R}$) is zero. Then (K,b) is isomorphic to a direct sum of hyperbolic lattices; in particular d is even. That is, there exists a basis $\{x_1, \dots, x_{d/2}, y_1, \dots, y_{d/2}\}$ of K such that*

$$(x_i, x_j) = (y_i, y_j) = 0, \quad (x_i, y_j) = \delta_{ij}.$$

See [Mil61].

This lemma on quadratic forms allows us to do surgery to kill K. Namely, choose a basis as above of K, and let us explain how to kill $\mathbb{Z}x_1 \oplus \mathbb{Z}y_1$, for instance. Using the strong form of the Whitney imbedding theorem ([Mil61]), choose an imbedding

$$\alpha : S^m \to M$$

representing x_1 (or rather, its Poincaré dual), as well as a nullhomotopy of $f \circ \alpha : S^m \to M \to X$. Since $(x_1, x_1) = 0$, we find that the normal bundle of α is trivial.

In particular, we can choose an imbedding $S^m \times D^m \hookrightarrow M$ extending α. Such an imbedding is equivalent (by the tubular neighborhood theorem) to a choice of trivialization of the normal bundle. The trivializations of the normal bundle of α are parametrized by $\pi_m(SO(m))$, while the trivializations of the *stable* normal bundle are parametrized by $\pi_m(SO)$. Now, the map

$$\pi_m(SO(m)) \to \pi_m(SO(m+1)) \twoheadrightarrow \pi_m(SO)$$

is a surjection. In fact, the cokernel of the first map injects into $\ker(\pi_m(S^m) \to \pi_{m-1}(SO(m))$, which is trivial since it sends the generator to the class tangent bundle of S^m (whose Euler class is nontrivial). Therefore, we can choose a trivialization of the normal bundle of α which is *compatible* with the trivialization of $f^*\xi$ given by the nullhomotopy of $f \circ \alpha : S^n \to X$.

In particular, we can do surgery, just as we could before, to get a map

$$f' : M' \to X, \quad M' = M \setminus (S^m \times \mathrm{Int} D^m) \cup_{S^m \times S^{m-1}} (D^{m+1} \times S^{m-1})$$

It is not obvious that the map f' is still m-connected and that we have actually simplified the relative homotopy groups. To see this, we consider the intermediate manifold-with-boundary $M_0 = M \setminus (S^m \times \mathrm{Int}(D^m))$. This is homotopy equivalent to $M \setminus (S^m \times \{0\})$ and to $M' \setminus (\{0\} \times S^{m-1})$, which interpolates between M and M'.

Fix a field k and take coefficients in k. We have a long exact sequence in cohomology

$$\dots \to H^{m-1}(M) \to H^{m-1}(M_0) \to H^m(M, M_0) \to H^m(M) \to H^m(M_0) \to \dots .$$

Observe that $H^m(M, M_0) \simeq H^m(S^m \times D^m, S^m \times (D^m \setminus \{0\})) \simeq k$ and the lower-dimensional cohomology groups of M, M_0 vanish. The map

$$H^m(M, M_0) \simeq k \to H^m(M)$$

can be identified with the map $1 \mapsto x_1$. We find that it is an injection. It follows that:

(1) $H^i(M) \to H^i(M_0)$ is an isomorphism for $i < m$.
(2) There is an exact sequence

$$H^m(M, M_0) \to H^m(M) \to H^m(M_0) \to 0,$$

which shows that $H^m(M_0)$ is a quotient space of $H_m(M)$, by the image of x_1.

We now play the same game with $M_0 \to M'$. Observe that the inclusion $M_0 \to M'$ is homotopy equivalent to $M' \setminus (\{0\} \times S^{m-1}) \to M'$, as observed above. Consequently,

$$H^*(M', M_0) \simeq H^*(D^{m+1} \times S^{m-1}, (D^{m+1} \setminus \{0\}) \times S^{m-1})).$$

This gives an exact sequence

$$\dots \to 0 \to H^m(M') \to H^m(M_0) \to H^{m+1}(M', M_0) \simeq k,$$

which shows that $H^i(M_0) \to H^i(M')$ is an isomorphism for $i < m$. Moreover, we find that $H^m(M') \subset H^m(M_0)$, so it is a k-vector space is of dimension at most that of $H^m(M_0)/kx_1$. In fact, the commutative diagram:

$$\begin{array}{ccccc} H^m(D^{m+1} \times S^{m-1}) & \longrightarrow & H^m((D^{m+1} \setminus \{0\}) \times S^{m-1}) & \longrightarrow & H^{m+1}(D^{m+1} \times S^{m-1}, (D^{m+1} \setminus \{0\}) \\ \uparrow & & \uparrow & & \uparrow \\ H^m(M') & \longrightarrow & H^m(M_0) & \longrightarrow & H^{m+1}(M', M_0) \end{array}$$

shows that the map $H^m(M_0) \to H^{m+1}(M', M_0)$ can be identified with pull-back along $\alpha : S^m \to M_0$. In particular, it maps y_1 to 1 and consequently $H^m(M)$ is *strictly* smaller than $H^m(M_0)$. (We don't actually need to know this, though.)

What does this tell us? Starting with a degree one normal map $f : M \to X$ which was m-connected, we have produced (by surgery on f) a new degree one normal map $f' : M' \to X$ with the following properties.

(1) The rank of $H_m(M')$ is smaller than that of $H_m(M)$. In particular, the kernel $K(f')$ is smaller than $K(f)$.
(2) $f' : M' \to X$ is still m-connected. To see this, observe that we had a zig-zag of isomorphisms between $H_i(M)$ and $H_i(M')$ for $i < m$. Namely, we have a commutative diagram:

$$\begin{array}{ccc} H_i(M_0) & \xrightarrow{\simeq} & H_i(M) \\ \downarrow \simeq & & f_* \downarrow \simeq \\ H_i(M') & \xrightarrow{f'_*} & H_i(X) \end{array}$$

where the map $f_* : H_i(M) \to H_i(X)$ is an isomorphism because f is m-connected. This diagram shows that $f'_* : H_i(M') \to H_i(X)$ is an isomorphism for $i < m$. For $i = m$, the map f'_* is already a surjection because f' is a normal map.

Repeating this process, we can keep applying surgery to get a manifold homotopy equivalent to X. □

References

[Ati61] M. F. Atiyah. Thom complexes. *Proc. London Math. Soc. (3)*, 11:291–310, 1961.

[Bro95] William Browder. Homotopy type of differentiable manifolds. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 97–100. Cambridge Univ. Press, Cambridge, 1995.

[Ker60] Michel A. Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960.

[KM63] Michel A. Kervaire and John W. Milnor. Groups of homotopy spheres. I. *Ann. of Math. (2)*, 77:504–537, 1963.

[Mil61] John Milnor. A procedure for killing homotopy groups of differentiable manifolds. In *Proc. Sympos. Pure Math., Vol. III*, pages 39–55. American Mathematical Society, Providence, R.I, 1961.

[MK60] John W. Milnor and Michel A. Kervaire. Bernoulli numbers, homotopy groups, and a theorem of Rohlin. In *Proc. Internat. Congress Math. 1958*, pages 454–458. Cambridge Univ. Press, New York, 1960.

[MM79] Ib Madsen and R. James Milgram. *The classifying spaces for surgery and cobordism of manifolds*, volume 92 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1979.

[Ran02] Andrew Ranicki. *Algebraic and geometric surgery*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2002. Oxford Science Publications.