ÉTALE π_1 OF A SMOOTH CURVE

AKHIL MATHEW

1. Introduction

One of the early achievements of Grothendieck's theory of schemes was the (partial) computation of the étale fundamental group of a smooth projective curve in characteristic p. The result is that if X_0 is a curve of genus g over an algebraically closed field of characteristic p, then its fundamental group is topologically generated by $2g$ generators. This is analogous to the characteristic zero case, where the topological fundamental group generated by $2g$ generators (subject to a single relation) by the theory of surfaces.

To motivate étale π_1, let's recall the following statement:

Theorem 1.1. *Let X be a nice topological space, and $x_0 \in X$. Then the functor $p^{-1}(x_0)$ establishes an equivalence of categories between covering spaces $p : \overline{X} \to X$ and $\pi_1(X, x_0)$-sets.*

This is one way of phrasing the Galois correspondence between subgroups of $\pi_1(X, x_0)$ and connected covering spaces of X, but which happens to be more categorical and generalizable. The interpretation of $\pi_1(X, x_0)$ as classifying covering spaces is ultimately the one that will work in an algebraic context. One can't talk about homotopy classes of loops in an algebraic variety. However, Grothendieck showed:

Theorem 1.2 ([1]). *Let X be a connected scheme, and $\overline{x_0} \to X$ a geometric point. Then there is a unique profinite group $\pi_1(X, \overline{x_0})$ such that the fiber functor of liftings $\overline{x_0} \to \overline{X}$ establishes an equivalence of categories between (finite) étale covers $p : \overline{X} \to X$ and finite continuous $\pi_1(X, x_0)$-sets.*

The intuition is that étale covers are supposed to be analogous to covering spaces (because, for complex varieties, étale morphisms are local isomorphisms on the analytifications). However, one wishes to retain finite type hypotheses in algebraic geometry, so it makes sense to classify *finite* étale covers. (Hereafter, the word étale cover will implicitly mean finite.)

I don't actually want to get into the proof of this theorem! Grothendieck did it by inventing his own formalism for Galois theory, which encompasses both classical Galois theory (of either field extensions or covering spaces) and the étale π_1. The idea is that any suitably nice (by which one means artinian, and satisfying some other conditions) category with an appropriate "fiber functor" F to the category of finite sets (here, the pre-images of one point) becomes equivalent (via F) to the category of finite continuous G-sets for G a uniquely determined profinite group. In fact, G is the group of automorphisms of the fiber functor. It is also the projective limit of the automorphism groups of *Galois* objects in this category.

Moreover, it turns out that, when one is working over the complex numbers, the étale fundamental group turns out to be the profinite completion of the usual fundamental group. This is one way of stating the so-called Riemann existence theorem:

Date: June 22, 2011.

Theorem 1.3. *Let X be a scheme of finite type over* $\mathrm{Spec}\mathbb{C}$. *Then the analytification functor establishes an equivalence of categories between the category of finite étale covers of X and the category of finite covering spaces of the analytification.*

In other words, we can use familiar transcendental techniques to compute the *étale* fundamental group of something in characterstic zero. The more interesting case is characteristic p. A proof is in [1], for instance.

Anyway, let's accept this, and try to sketch the idea behind Grothendieck's proof. Fix a smooth curve X_0 over a field of characteristic p. The first step is to *lift* X_0 to characteristic zero. The existence of the lifting is not trivial, but let's assume it exists for now. In other words, choose a complete discrete valuation ring A of characteristic zero whose residue field is the field of characteristic p and a proper, smooth scheme $X \to \mathrm{Spec}A$ such that the special fiber is X_0.

Then, one can, using some of Grothendieck's scheme-theoretic machinery, find relations between the fundamental groups of the special and general fiber. In particular, one can define a surjective "cospecialization" map between the general and specific fiber. But the general fiber is a curve in characteristic zero, and it has the same genus as X_0 by flatness of $X \to \mathrm{Spec}A$, so the usual transcendental methods allow one to compute π_1 of the general fiber. This will be the strategy.

The purpose of this note is to give a detailed exposition of this theorem of Grothendieck. For the most part, we follow [1] in the exposition.

2. Lifting curves to characteristic zero

It is a general question of when one can "lift" varieties in characteristic p to characteristic zero. Doing so often allows one to bring in transcendental techniques (to the lift), as it will in this case of π_1. Let us thus be formal:

Definition 2.1. Let X_0 be a proper, smooth scheme of finite type over a field k of characteristic p. We say that a **lifting** of X_0 is the data of a DVR A of characteristic zero with residue field k, and a proper, smooth morphism $X \to \mathrm{Spec}A$ whose special fiber is isomorphic to X_0.

There are obstructions that can prevent one from making such a lifting. One example is given by étale cohomology. A combination of the so-called proper and smooth base change theorems (see [**?**]) implies that, in such a situation, the cohomology of the special fiber and the cohomology of the general fiber, with coefficients in any finite group without p-torsion, are isomorphic. As a result, if there is something funny in the étale cohomology of X_0, it might not be liftable.

In the case of curves, fortunately, it turns out there are no such problems, but still actually lifting one will take some work. We aim to prove:

Theorem 2.2. *Let $X_0 \to \mathrm{Spec}k$ be a smooth, proper curve of finite type over the field k of characteristic p. Then if A is any complete DVR of characteristic zero with residue field k, there is a smooth lifting $X \to \mathrm{Spec}A$ of X_0.*

One should, of course, actually check that such a complete DVR does exist. But this is a general piece of algebra, found for instance in [4].

The reason there won't be any obstructions in the case of curves is that they are of dimension one, but we'll see that the cohomological obstructions to lifting all live in H^2. The strategy, in fact, will be to lift $X_0 \to \mathrm{Spec}k$ to a sequence of smooth schemes $X_n \to \mathrm{Spec}A/\mathfrak{m}^n$ (where $\mathfrak{m} \subset A$ is the maximal ideal) that each lift each other, using the local nilpotent lifting property of smooth morphisms. This family $\{X_n \to \mathrm{Spec}A/\mathfrak{m}^n\}$ is an example of a so-called *formal scheme,* which for our purposes is just such a compatible sequence of liftings. Obviously any scheme $X \to \mathrm{Spec}A$ gives rise to a formal scheme (take the base-changes to $\mathrm{Spec} A/\mathfrak{m}^n$), but it is actually nontrivial (i.e., not always true) to show that a formal scheme is indeed of this form. But we will be able to do this as well in the case of curves.

2.1. **Local lifting.** Let us start by lifting the smooth curve X_0 to a sequence of schemes $X_n \to \mathrm{Spec} A/\mathfrak{m}^n$, following the program outlined earlier. It will be convenient to do this in a more general setting. Let S be a base scheme, and let $S_0 \subset S$ be a subscheme defined by an ideal $\mathcal{I}$ of square zero. Suppose $X_0 \to S_0$ is a smooth scheme. We want to know if there is a smooth scheme $X \to S$ whose restriction to S_0 is S. In general, this need not exist, but the next result states that the smooth lifting does *locally.*

Hereafter, all schemes are noetherian.

Proposition 2.3 (Local lifting of smooth morphisms)**.** *Lifting $X_0 \to S_0$ to a smooth S-scheme can always be done locally. If $x_0 \in X_0$, there is a neighborhood $U_0 \subset X_0$ of x_0, a smooth scheme $U \to S$ such that $U \times_S S_0 \simeq U_0$.*

Moreover, any two liftings are locally S-isomorphic. If S is affine, and U, U' lift the open affine U_0, there is an S-isomorphism $U \simeq U'$.

Of course, it is not very deep to lift the schemes themselves: the composite $X_0 \to S_0 \to S$ would do. The point is to preserve essential properties (in this case, smoothness).

Proof. We are going to deduce it from the "équivalence remarkable de catégories" of Grothendieck, that states the following: if $S_0 \subset S$ is a closed subscheme defined by an ideal of square zero, then base-change gives an *equivalence of categories* between the collection of schemes étale over S and the schemes étale over S_0. In other words, étale S_0-schemes can be lifted globally (and uniquely). For smooth morphisms the statement is weaker.

Indeed, we note that (by one characterization of smoothness) there is a neighborhood U_0 of x_0 such that the map

$$U_0 \to S_0$$

factors as a composite

$$U_0 \xrightarrow{g} \mathbb{A}^n_{S_0} \to S_0$$

where g is étale. Now U_0 then extends *uniquely* to a scheme U étale over $\mathbb{A}^n_S$. This is the lifting we want.

Finally we need to show local uniqueness of the lifting. We will do this using the *infinitesimal lifting property.* Let U_0 be affine, $U_0 = \mathrm{Spec} B_0$. Suppose S and S_0 are affine, without loss of generality, say $S = \mathrm{Spec} A, S_0 = \mathrm{Spec} A_0 = \mathrm{Spec} A/I$ where $I \subset A$ is an ideal of square zero. By hypothesis, we are given two smooth A-algebras B_a, B_b (whose spectra are the two liftings U, U'), together with an isomorphism

$$B_a/IB_a \simeq B_b/IB_b \simeq B_0.$$

But here we use the infinitesimal lifting property of smooth morphisms. Namely, the map

$$B_a \to B_b \otimes_A A/I$$

can be lifted to an A-homomorphism $B_a \to B_b$ because B_a is A-smooth. This homomorphism, moreover, induces the identity mod I (when both are identified with B_0). The claim is that this is an isomorphism, which follows from the next lemma. □

Lemma 2.4. *Let A be a noetherian ring. Let B, B' be flat A-algebras, and let $I \subset A$ be an ideal of square zero. If a map $f : B \to B'$ is such that the reduction mod I, $B \otimes_A A/I \to B' \otimes_A A/I$, is an isomorphism, then f is itself an isomorphism.*

Proof. Indeed, we note that by flatness, $(I/I^2) \otimes_A B = (I/I^2) \otimes_{A/I} (B/IB) = IB/I^2B = IB$. Similarly for B'. That is, flatness makes the associated graded behave nicely. But, again by flatness:

$$I \otimes_A B = IB, \quad I \otimes_A B' = IB'$$

and consequently the map $IB \to IB'$ is an isomorphism. In particular, the map $B \to B'$ induces an isomorphism on the associated gradeds of the I-adic filtration. Since I is nilpotent, this gives the result. □

The point of the lemma is that $B/IB, B'/IB'$ determine the associated gradeds by flatness.

In the proof of the above result, we showed something more specific than just local unicity. When the base and the target are *affine*, then any two smooth liftings are isomorphic (noncanonically, in general).

2.2. **Global lifting.** So we can always lift smooth things locally. Of course, there will be lots of ways of doing that in general, and the question is whether we can patch them together. In the étale case, there are no nontrivial automorphisms of the lifting: that is, the functor of base-changing by S_0 is fully faithful (by the infinitesimal lifting property for étale morphisms). As a result, there is no problem patching local liftings.

For smooth morphisms, the problem is more delicate. We can always lift locally, but to patch the liftings one needs a "cocycle" condition, which is not automatic. As a result, there is a cohomological obstruction to lifting that comes from these automorphisms.

Let X_0 be a smooth S_0-scheme. Suppose $S_0 \hookrightarrow S$ as a subscheme defined by an ideal of square zero. We are going to define a *sheaf* $\mathcal{F}$ on X_0 as follows: for each $U_0 \subset X_0$, $\mathcal{F}(U_0)$ will denote the set of S-automorphisms $U \to U$ (where U is the pre-image of U_0 in X) inducing the identity on U_0.

We shall use:

Lemma 2.5. *The sheaf $\mathcal{F}$ is canonically isomorphic to a quasi-coherent sheaf on X_0, which is independent of the lifting X.*

This is a remarkable statement. It is clear that $\mathcal{F}$ can be made into a *sheaf* of groups (since $S_0 \subset S$ is defined by an ideal of square zero, it is easy to see that an automorphism of U restricts to an automorphism of V). In fact it is not even obvious a priori that this sheaf of groups is a sheaf of abelian groups.

Proof. To see this, we will start by assuming X_0, S_0 (and consequently X) affine. Say $S_0 = \mathrm{Spec} A_0 = \mathrm{Spec} A/I, S = \mathrm{Spec} A, X = \mathrm{Spec} B, X_0 = \mathrm{Spec} B_0 = \mathrm{Spec} B/IB$. Then we are looking for the set of A-homomorphisms

$$f : B \to B$$

that induce the identity $B/IB \to B/IB$. Such a map is necessarily of the form $1+\phi$, where $\phi : B \to IB$ is an A-homomorphism. One requires, of course:

$$(b + \phi(b))(b' + \phi(b')) = bb' + \phi(bb').$$

Since $I^2 = 0$, this is equivalent to saying that ϕ is an A-*derivation* $B \to IB$.

One can check that the composite $(1+\phi)\circ(1+\phi') = 1+\phi+\phi'+\phi\circ\phi' = 1+\phi+\phi'$, as the composite of two A-derivations $B \to IB$, $B \to IB$ is always zero. As a result, the *group* of such automorphisms is isomorphic to the group of derivations.

Such derivations are classified by maps of B-modules

$$\Omega_{B/A} \to IB.$$

However, these are the same as maps of B_0-modules $\Omega_{B_0/A_0} \to IB$ because IB is a B_0-module, as $I^2 = 0$. It follows that the sheaf in question is the sheaf

$$\mathcal{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/S_0}, \mathcal{I}\mathcal{O}_X)$$

where $\mathcal{I}$ is the ideal of square zero cut that cuts out S_0. Note that $\mathcal{I}\mathcal{O}_X$ is an $\mathcal{O}_{X_0}$-module. This is coherent and completely independent of X: indeed, it is also isomorphic to

$$T_{X_0/S_0} \otimes_{\mathcal{O}_{S_0}} \mathcal{I}_{S_0}$$

where T_{X_0/S_0} is the dual of the (locally free) cotangent sheaf and $\mathcal{I}_{S_0}$ is the ideal of S_0 in S. This depends on the embedding $S_0 \to S$ but not the lifting X. □

Now let us try to analyze the situation we are ultimately interested in. Let $X_0 \to S_0$ be a smooth separated scheme, where $S_0 \hookrightarrow S$ is a subscheme defined by an ideal of square zero. For simplicity, let us assume S_0, S affine, since that is the only case we shall need. We know that there is an open affine cover $\{U_0^\alpha\}$ of X_0 consisting of schemes that lift to smooth affine S-schemes $\{U^\alpha\}$ such that

$$U_0^\alpha = U^\alpha \times_S S_0.$$

Now, for each α, β, we have two liftings of the *affine* scheme $U_0^\alpha \cap U_0^\beta$: namely, $U^{\alpha,\beta} \subset U^\alpha$ and $U^{\beta,\alpha} \subset U^\beta$. By *formal* smoothness (namely, the infinitesimal lifting property), we have isomorphisms

$$t^{\alpha\beta} : U^{\alpha,\beta} \simeq U^{\beta,\alpha}$$

that induce the identity $U_0^\alpha \cap U_\beta^0 \to U_0^\alpha \to U_0^\beta$. This is where we have used the affineness and separatedness hypotheses. The hope is that these would satisfy the cocycle condition, and that we could glue all the $\{U^\alpha\}$ together. Unfortunately, they needn't.

So let's recall the cocycle condition: for any three indices α, β, γ, one must have

$$t^{\beta\gamma} \circ t^{\alpha\beta} = t^{\alpha\gamma} : U^{\alpha\gamma} \cap U^{\alpha\beta} \to U^{\gamma\alpha} \cap U^{\gamma\beta}.$$

If we had this, then we could just glue the U^α and get a smooth lifting of X_0 to S.

What we can do is to consider the differences $(t^{\alpha\gamma})^{-1} \circ t^{\beta\gamma} \circ t^{\alpha\beta}$, which are automorphisms of $U^{\alpha\gamma} \cap U^{\alpha\beta}$. By the previous lemma, this is a 2-Cech cocycle with values in the sheaf $\mathcal{F}$ defined as above, over the open cover $\{U_0^\alpha\}$. (One should check that it is actually a 2- cocycle, but this is formal.)

Now, if it were a boundary, then we would be done and we could make the lifting. For if we have a 1-cochain in $\mathcal{F}$, this would be a collection of automorphisms $q^{\alpha,\beta}$ of each $U^{\alpha\beta}$ such that, for each triple (α, β, γ), one has:

$$(t^{\alpha\gamma})^{-1} \circ t^{\beta\gamma} \circ t^{\alpha\beta} = q^{\alpha\beta} q^{\beta\gamma} (q^{\alpha\gamma})^{-1}.$$

The point is now that if this 2-cocycle is a coboundary, then we can use the $q^{\alpha\beta}$ to *modify* the transition maps $t^{\alpha\beta}$ (by, say, precomposition) so as to have satisfied the cocycle condition. In particular, if the Cech H^2 of this sheaf vanishes with respect to the open cover U_0^α, the lifting exists.

Note that this is a good Cech cover of X_0 because X_0 is separated and eaech U_0^α is affine. It follows that if X_0 is of dimension one and $H^2(X_0, \mathcal{F}) = 0$, then the lifting exists.

Corollary 2.6. *Let $X_0 \to k$ be a smooth curve. Let A be a DVR with residue field k and maximal ideal $\mathfrak{m}$. Then there is a compatible system of smooth schemes $X_n \to \mathrm{Spec} A/\mathfrak{m}^n$.*

2.3. **From formal to actual.** Let $X_0 \to k$ be a smooth curve. If A is a complete DVR with residue field k, then we have seen how to lift $X_0 \to k$ to a sequence $X_n \to \mathrm{Spec} A/\mathfrak{m}^n$ of compatible smooth schemes by the previous section. Namely, we first lift to $\mathrm{Spec} A/\mathfrak{m}^2$, lift that to $\mathrm{Spec} A/\mathfrak{m}^3$, and continue repeatedly. This is still rather far from our ultimate goal, which is a smooth, proper scheme $X \to \mathrm{Spec} A$.

Now, with the mechanics of the lifting procedure behind us, we want to turn the system of schemes $X_n \to \mathrm{Spec} A/\mathfrak{m}^n$ into an actual scheme. We have already stated that this is a so-called *formal scheme*, which we will denote by the symbol $\mathfrak{X}$, and write formally

$$\mathfrak{X} \to \mathrm{Sppf} A,$$

to indicate the system of maps $X_n \to \mathrm{Spec} A_n$. We shall think of formal schemes very naively; we do not need to worry about their general theory, so shall treat them as a black box here. We shall write $\mathfrak{X} \times_A k = X_0$, and similarly for $\mathfrak{X} \times_A A/\mathfrak{m}^n$ for X_n.

We now that one way of obtaining a formal scheme is to start with an actual scheme $X \to \mathrm{Spec} A$ and consider each of the reductions mod $\mathfrak{m}^n$. Such formal schemes are said to be **algebrizable.** This is the formal analog of complex analytic spaces that come from algebraic varieties. There are formal schemes, even proper ones, that are not algebrizable, so we are going to need a special tool in the case of curves.

That tool is:

Theorem 2.7 ([2])**.** *Let A be a complete local ring, $\mathfrak{X} \to \mathrm{Sppf} A$ a formal scheme. Suppose $\mathfrak{X} \times_A k$ is proper. Suppose moreover there is a compatible system of line bundles $\mathcal{L}_n$ on each $X_n = \mathfrak{X} \times_A A/\mathfrak{m}^n$ such that $\mathcal{L}_0$ is very ample on $X_0 \to k$.*

Then there is a projective morphism $X \to \mathrm{Spec} A$, such that the "formal scheme" $\mathfrak{X}$ is obtained from X: that is, $\mathfrak{X}$ is algebrizable.

This is a consequence of Grothendieck's "formal GAGA" and appears in EGA III.5. We shall not prove this.

But we shall use it. Consider a formal scheme $\mathfrak{X} \to \mathrm{Sppf} A$ obtained by successively lifting a smooth proper curve X_0 over the residue field $A/\mathfrak{m}$. Now X_0 is projective, so it has a very ample line bundle on it. If we can lift this to each $X_n \to \mathrm{Spec} A/\mathfrak{m}^n$, then the above result will imply that the formal scheme is algebrizable, and then we will have lifted X_0 to characteristic zero.

Here the fact that we are in dimension one saves us (again!), because we can successively lift the ample line bundle step by step. We use:

Lemma 2.8. *Let X be a scheme of dimension one, $X_0 \subset X$ a closed subscheme defined by an ideal of square zero. Then the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ is surjective.*

Proof. Suppose X_0 is defined by the ideal $\mathcal{I}$ of square zero. There is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where the first map sends $x \mapsto 1 + x$. This is a general fact about rings, even. Now since $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ (and similarly for X_0), the long exact sequence in cohomology and $H^2(X, \mathcal{I}) = 0$ gives the result. □

It follows that we can lift the sequence of smooth schemes $\{X_n \to \mathrm{Spec} A/\mathfrak{m}^n\}$ to a projective scheme $X \to \mathrm{Spec} A$. The only thing left to see is that X is smooth over $\mathrm{Spec} A$. This follows because it is smooth on the special fiber: indeed, one checks flatness by the infinitesimal criterion. It is a general fact that if $f : X \to Y$ is a finite-type morphism of noetherian schemes, and $x \in X$ is such that the fiber $X_{f(x)}$ is smooth over $k(f(x))$ and f is flat at x, then f is smooth at x. With this in mind, it is clear that $X \to \mathrm{Spec}A$ is smooth on the specific fiber.

But this means in particular that X is smooth everywhere! Indeed, the smooth locus of a morphism is always open. Let $T \subset X$ be the collection of points where f is not smooth. Then the image of T is closed because X is proper over $\mathrm{Spec}A$, but this image does not contain the closed point; as a result, it is empty. So X is A-smooth.

This completes the proof that smooth curves can be lifted to characteristic zero.

3. The theory of specialization of π_1

Let $X_0 \to \mathrm{Spec}k$ be a smooth curve over an algebraically closed field k of characteristic p. We are interested in determining a set of topological generators for this curve. To do this, we started by showing that if A is a complete DVR with residue field k, then one can "lift" (by using cohomological vanishing and formal-to-algebraic comparison theorems) $X_0 \to \mathrm{Spec}k$ to a smooth, proper morphism $X \to \mathrm{Spec}A$. Ideally, we will have chosen A to be characteristic zero itself. Now our plan is to compare the two geometric fibers of X: one is X_0, and the other is $X_{\overline{\xi}}$ (where ξ is the generic point; the over-line indicates that one wishes an algebraic closure of $k(\xi) = K(A)$ here) with each other. Ultimately, we are going to show two things:

1. The natural map $\pi_1(X_0) \to \pi_1(X)$ is an isomorphism.
2. The natural map $\pi_1(X_{\overline{\xi}}) \to \pi_1(X)$ is an *epimorphism.*

Here we have been loose with notation, as we have not indicated the relevant geometric points. The geometric point is, however, irrelevant for a connected scheme.

It will follow from this that there is an continuous epimorphism of profinite groups

$$\pi_1(X_{\overline{\xi}}) \to \pi_1(X_0).$$

However, $\pi_1(X_{\overline{\xi}})$ will be seen to be topologically generated by $2g$ generators (where g is the genus) by comparison with a curve over $\mathbb{C}$. For a smooth curve of genus g over $\mathbb{C}$, it is clear from the Riemann existence theorem (and the topological fundamental group) that π_1 has $2g$ topological generators.

Thus, it will follow, as stated earlier:

Theorem 3.1. *If X_0 is a smooth curve of genus g over an algebraically closed field of any characteristic, $\pi_1(X_0)$ is topologically generated by $2g$ generators.*

One technical point will be, of course, that it is not entirely obvious that $\pi_1(X_{\overline{\xi}})$ is the same as it would be for a curve over $\mathbb{C}$. This requires independent proof, but it will not be too hard.

3.1. The specific fiber.

Let $(A, \mathfrak{m})$ be a complete local noetherian ring, and let $X \to \mathrm{Spec}A$ be a proper morphism. There is a "specific" fiber $X_0 \to \mathrm{Spec}k$ of X, where $k = A/\mathfrak{m}$ is the residue field of A.

Let $\overline{x} \to X_0$ be a geometric point. Since there is a natural inclusion $X_0 \to X$, there is a natural map $\pi_1(X_0, \overline{x}) \to \pi_1(X, \overline{x})$. Recall the definition of this map. For a scheme Y, let $\mathrm{Et}(Y)$ denote the category of schemes finite and étale over Y (i.e. étale covers). Then there is a functor (base-change)

$$\mathrm{Et}(X) \to \mathrm{Et}(X_0).$$

Note that $\mathrm{Et}(X)$ is identified with the category of finite continuous $\pi_1(X, \overline{x})$-sets, and $\mathrm{Et}(X_0)$ with the category of finite continuous $\pi_1(X_0, \overline{x})$-sets. The map

$$\pi_1(X_0, \overline{x}) \to \pi_1(X, \overline{x})$$

is the one inducing the corresponding functor from $\pi_1(X, \overline{x})$-sets to $\pi_1(X_0, \overline{x})$-sets.

We want to show that this is an isomorphism. This will follow directly as a consequence of the next result.

Theorem 3.2. *If $X \to \mathrm{Spec}A$ is proper for A local, complete, and noetherian, and X_0 is the specific fiber, then the functor of base-change induces an equivalence of categories between* $\mathrm{Et}(X)$ *and* $\mathrm{Et}(X_0)$.

This is a generalization of the more elementary fact that if A is a complete local noetherian ring with residue field k, then the functor $B \mapsto B \otimes_A k$ induces an equivalence of categories between finite étale A-algebras and finite étale k-algebras.

Proof. Let us first start by showing that it is fully faithful. Let Y, Y' be étale X-schemes. Then a X-morphism $Y \to Y'$ can be described as a certain subscheme of the product $Y \times_X Y'$, via its graph Γ. Namely, this subscheme Γ must be open and closed, and of degree one over Y.

To see this, note that there is a cartesian diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & Y \times_X Y' \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y' \times_X Y'. \end{array}$$

However, Y' is unramified over X, so the diagonal map $Y' \to Y' \times_X Y'$ is an open immersion. It is also separated (being finite), so the diagonal map is a closed immersion. Thus Γ is a union of connected components of $Y \times_X Y'$. Moreover, any such open and closed $\Gamma \subset Y \times_X Y'$ gives rise to a morphism $Y \to Y'$ if the projection $\Gamma \to Y$ is an isomorphism, which is equivalent to saying that it has degree one (because all these morphisms are étale). The same can be said for morphisms $Y_0 \to Y_0'$.

However, we will see below that open and closed subsets of $Y \times_X Y'$ in natural bijection with open and closed subsets of $Y_0 \times_{X_0} Y_0'$; moreover, it is clear that the degree of such a subset $\Gamma \subset Y \times_X Y'$ is one over Y if and only if the degree of Γ_0 over Y_0 is one. (This follows again because Y is connected if and only if Y_0 is, which we will see below.)

Now let us show that the functor is essentially surjective. That is, given an étale cover $Y_0 \to X_0$, we need to find an étale cover $Y \to X$ whose fiber is $Y_0 \to X_0$. To do this, we start by considering the family of infinitesimal thickenings $X_n = X \times_A A/\mathfrak{m}^{n+1}$. We know that Y_0 can be lifted uniquely to a series of étale covers $Y_n \to X_n$, by the "équivalence remarquable de catégories" that any scheme étale over another scheme lifts uniquely to a nilpotent thickening (and in a fully faithful way[1]). Note that said equivalence is not restricted to étale covers.

So given $Y_0 \to X_0$, we get a family of finite étale covers $Y_n \to X_n$. In other words, we get a *formal scheme* $\mathfrak{Y} \to \mathfrak{X}$, where $\mathfrak{X}$ is identified with the family $\{X_n\}$. We want to know now that $\mathfrak{Y}$ is algebrizable, or that it comes from a proper scheme $Y \to X$.

To see this, we will use Grothendieck's *existence theorem* in formal geometry. This theorem states:

Theorem 3.3 (Formal GAGA, [2])**.** *If $X \to \mathrm{Spec}A$ is proper and A complete, local, and noetherian, then there is an equivalence of categories between coherent sheaves on X and coherent sheaves on the formal completion $\mathfrak{X}$.*

We shall not prove this. In fact, we have not even properly defined a *formal completion,* let alone what a coherent sheaf on a formal scheme is. For our purposes, it suffices to say that the formal completion is the family of schemes $\left\{X_n \to \mathrm{Spec}A/\mathfrak{m}^{n+1}\right\}$ (note that each X_n is the reduction of

[1]This is a very important fact, and follows from a combination of the nilpotent lifting property of étale extensions as well as the local structure.

X_{n+1}) and a coherent sheaf on the formal completion is a compatible family of coherent sheaves on the $\{X_n\}$. The point is that this compatible family has to come from a well-defined coherent sheaf on X.

The analogous ordinary GAGA states that, on a proper variety over $\mathbb{C}$, analytification establishes an equivalence of categories between the category of coherent sheaves on X and coherent analytic sheaves on the analytification. As with formal GAGA, the hard part is not the full faithfulness of the functor, but the fact that any analytic sheaf comes from an algebraic one. The proof of ordinary GAGA is strikingly similar to that of formal GAGA, however. (It is perhaps more correct to say that sentence in the reverse order: formal GAGA came after Serre's paper!)

Anyway, let's return to the main goal: if one has a *formal* étale cover $\mathfrak{Y} \to \mathfrak{X}$—that is, a compatible family of étale covers $\{Y_n \to X_n\}$—then the one compatible family of coherent sheaves on the $\{X_n\}$, because each Y_n is a **Spec** of a coherent $\mathcal{O}_{X_n}$-algebra, say $\mathcal{F}_n$. This family $\{\mathcal{F}_n\}$ must come from a coherent sheaf $\mathcal{F}$ on X, and since each $\mathcal{F}_n$ is an *algebra*, it follows again formally that we can lift the algebra structure to $\mathcal{F}$. We get a finite map

$$Y = \mathbf{Spec}\mathcal{F} \to X.$$

If we can show that this is étale, then we will be done.

But étaleness now follows formally, as before: namely, by the infinitesimal criterion of flatness and étaleness of $Y_0 \to X_0$, we find that $Y \to X$ is étale at all points in the special fiber. By properness, as before, it follows that $Y \to X$ is étale everywhere, since the étale locus is open. This proves the essential surjectivity.

□

Finally, to complete the proof of the above theorem, we need to prove:

Lemma 3.4. *Let X be a proper scheme over* $\mathrm{Spec}A$*, where A is a complete local noetherian ring, and let $X_0 = X \times_A k$. Then the map $\pi_0(X_0) \to \pi_0(X)$ is an isomorphism.*

Here π_0 denotes the functor of connected components.

Proof. Indeed, this follows from the formal function theorem, since open and closed subsets of *any* locally ringed space $(X, \mathcal{O}_X)$ are in bijection with idempotents in the ring $\Gamma(X, \mathcal{O}_X)$.

Note that we can consider the infinitesimal liftings $X_n = X \times_A A/\mathfrak{m}^{n+1}$ for each n. Each of these has the same topological space as X_0, and in particular the same π_0.

However, by the formal function theorem we have

$$\Gamma(X, \mathcal{O}_X) = \varprojlim \Gamma(X_n, \mathcal{O}_{X_n}).$$

In particular, the *idempotents* in $\Gamma(X, \mathcal{O}_X)$ are in bijection with $\varprojlim \mathrm{Idem}\Gamma(X_n, \mathcal{O}_{X_n})$ where Idem is the functor that assigns to each ring its set of idempotents. (Idem is a corepresentable functor, so it commutes with projective limits.) Anyway, since $\mathrm{Idem}\Gamma(X_n, \mathcal{O}_{X_n}) = \Gamma(X_0, \mathcal{O}_{X_0})$, it follows that

$$\mathrm{Idem}\Gamma(X, \mathcal{O}_X) = \mathrm{Idem}\Gamma(X_0, \mathcal{O}_{X_0}).$$

By the connection between idempotents and π_0, this completes the proof. □

3.2. **The exact sequence.** Our next goal is to obtain a small analog of the classical long exact sequence of a fibration in homotopy theory. Namely, the smooth proper morphism $X \to \mathrm{Spec}A$ constructed as earlier that lifts the curve $X_0 \to \mathrm{Spec}k$ will be the "fibration," and we are going to take the (geometric) fiber over the generic point. Since the base $\mathrm{Spec}A$ has trivial π_1 (because A is complete local and its residue field is algebraically closed), it will follow from this long exact sequence that

$$\pi_1(X_{\bar{\xi}}) \to \pi_1(X)$$

is a surjection.

We shall start by proving an auxiliary result. To motivate it, recall that a proper morphism $f: X \to Y$ has a *Stein factorization* via $X \to \mathbf{Spec} f_* \mathcal{O}_X \to Y$ (note that the second morphism is finite because $f_* \mathcal{O}_X$ is coherent). This result states that the Stein factorizations of certain morphisms give a means of constructing étale covers.

Theorem 3.5. *Let $f : X \to Y$ be a proper, flat morphism of noetherian schemes with geometrically reduced fibers. Then the map $\mathbf{Spec} f_* \mathcal{O}_X \to Y$ is an étale cover.*

Note that $\mathbf{Spec} f_* \mathcal{O}_X \to Y$ is the second half of the Stein factorization of f. Since f is proper, it is automatic that this map is finite.

Proof (following [3]*).* We shall start by employing the standard Grothendieckian trick of reducing to the case where the base is the spectrum of a complete local noetherian ring (whose residue field is even algebraically closed).

Everything is local on Y. We may thus assume Y is affine, $Y = \mathrm{Spec} A$. Moreover, the construction of $f_* \mathcal{O}_X$ commutes with flat base change (in general, the higher direct images by a separated morphism commute with flat base change). In particular, the Stein factorization commutes with flat base change. Since a morphism (say, of finite type) $Z_1 \to Z_2$ is étale if and only if $Z_1 \times_{Z_2} \mathrm{Spec} \mathcal{O}_{z, Z_2} \to \mathrm{Spec} \mathcal{O}_{z, Z_2}$ is étale for each $z \in Z_2$, we may (in view of this observation) assume furthermore that A is noetherian *local.*

Moreover, since étaleness descends under faithfully flat base change, we can make a faithfully flat base extension to ensure that A's residue field is algebraically closed. (In fact, it is a general lemma that one may produce flat local noetherian extensions of a local noetherian ring with any residue field extension whatsoever.) In view of the faithfully flat morphism $A \to \hat{A}$, we can even assume A is complete local with an algebraically closed residue field.

So let A be complete local with residue field k (algebraically closed), and let $X \to \mathrm{Spec} A$ be a proper, flat morphism whose fibers are geometrically reduced. We shall show:

(1) $H^0(X, \mathcal{O}_X)$ is a flat and finitely generated (i.e. free) A-module.
(2) $H^0(X, \mathcal{O}_X) \otimes_A k$ is a product of copies of k.

Together, these will imply that $H^0(X, \mathcal{O}_X)$ is an étale A-algebra (because it is flat and unramified). The techniques are essentially those in the proof of "cohomology and base change." Indeed, this claim follows directly from that general result.

Let us consider the functor T from finitely generated A-modules to finitely generated A-modules,

$$T(M) = H^0(X, \mathcal{O}_X \otimes_A M).$$

Since $X \to \mathrm{Spec} A$ is flat, a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$ induces a short exact sequence $0 \to \mathcal{O}_X \otimes_A M' \to \mathcal{O}_X \otimes_A M \to \mathcal{O}_X \otimes_A M'' \to 0$. In particular, we see that T is *left exact.*

We will show:

Lemma 3.6. *T is right-exact, and indeed, isomorphic to the functor $M \mapsto T(A) \otimes_A M$.*

This will easily give the result that $H^0(X, \mathcal{O}_X)$ is flat, because T is also left-exact.

Proof. There is a natural homomorphism

$$T(A) \otimes_A M \to T(M).$$

This is given by sending $x \in T(A), m \in M$ to the image of x under the homomorphism $A \stackrel{1 \mapsto m}{\to} M$. We will show that this map is surjective. Denote the previous functor by $U(M)$; this is right-exact. The claim is that $U(M) \to T(M)$ is always surjective.

We shall first prove this assuming that A is *local artinian* with maximal ideal $\mathfrak{m} \subset A$. Let X_0 be the specific fiber $X \times_A k$. Then note that $U(k) \to T(k)$ is surjective. For $U(k) = H^0(X, \mathcal{O}_X) \otimes_A k$ and $T(k) = H^0(X_0, \mathcal{O}_{X_0})$. The latter is a free k-vector space on the connected components of X_0, since the geometric fibers of $X \to \mathrm{Spec} A$ are reduced. But the primitive idempotents are hit by elements of $H^0(X, \mathcal{O}_X)$ because the map $\pi_0(X) \to \pi_0(X_0)$ is an isomorphism.

In general, we note:

Lemma 3.7. *Let A be an artinian local ring with residue field k. Let T be an additive, half-exact functor from the category of finitely generated A-modules to itself. Suppose $T(A) \to T(k)$ is surjective. Then T is of the form $M \mapsto T(A) \otimes_A M$ and $T(A)$ is flat.*

Proof. Indeed, if we let $U(M) = T(A) \otimes_A M$ as before, then we have a natural transformation $U(M) \to T(M)$. We have seen that this is a surjection for $M = k$. In general, induct on the length of M. Suppose M is an A-module of length greater than one; there is then an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0,$$

where M_1, M_2 have smaller lengths. We can find an exact and commutative diagram:

$$\begin{array}{ccccccc} U(M_1) & \longrightarrow & U(M) & \longrightarrow & U(M_2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ T(M_1) & \longrightarrow & T(M) & \longrightarrow & T(M_2). & & \end{array}$$

The outside vertical maps are surjective. From this a diagram chase (or the snake lemma, if $T(M_1)$ is replaced by its image in $T(M)$) shows that $U(M) \to T(M)$ is surjective.

Finally, we want to see that $U(M) \to T(M)$ is *bijective.* We can see this, however, by noting now that both U and T are right-exact (for T is now right-exact, as the surjective image of U) functors on the category of A-modules; there is a natural transformation $U(M) \to T(M)$ that is an isomorphism for free A-modules. By the "finite presentation trick," (i.e. choosing a presentation $F' \to F \to M \to 0$ of any finitely generated A-module with F, F' free) it now follows that the natural transformation is an isomorphism for any finitely generated A-module M. □

In general, of course, the complete local ring A constructed earlier will not be artinian. However, the functor T will be extra-friendly, though: it will have the property that, for any finitely generated A-module M,

$$T(M) \to \varprojlim_n T(M/\mathfrak{m}^n M) \tag{1}$$

is an isomorphism. This is in fact a consequence of the formal function theorem and the completeness of A.

Now we use:

Lemma 3.8. *Let A be a complete local noetherian ring, T a half-exact functor from finitely generated A-modules to finitely generated A-modules such that* (1) *is always an isomorphism. Then T is right-exact, and so isomorphic to the functor $M \mapsto T(A) \otimes_A M$.*

If T is additionally *left*-exact, it follows that $T(A)$ must be flat.

Proof. It follows that the natural transformation

$$T(A) \otimes_A M \to T(M)$$

is an isomorphism for each finitely generated M annihilated by a power of $\mathfrak{m}$, by the previous lemma for the artinian rings $A/\mathfrak{m}^n$.

Now fix a general M. We then have a natural isomorphism

$$T(A) \otimes_A \varprojlim M/\mathfrak{m}^n M \to \varprojlim T(M/\mathfrak{m}^n M);$$

the left side is clearly isomorphic to $T(A) \otimes_A M$, and the right side to $T(M)$ (by the condition on (1)). The lemma is thus clear. □

So it follows that in our case, for $T(M) = H^0(X, \mathcal{O}_X \otimes_A M)$, this half-exact functor is actually right-exact (since $T(A) \to T(k)$ is surjective). Moreover $T(A)$ is a flat A-module, since the functor is left-exact. Finally, $T(k) = T(A) \otimes_A k$ is the ring of regular functions on a reduced proper k-variety, so it is a product of copies of k. This proves the result. □

□

Here is an important consequence of the general formalism developed above. If $f : X \to Y$ is a proper and flat morphism with geometrically reduced fibers, then the formation of $f_*(\mathcal{O}_X)$ commutes with base change. To be more precise, if one has a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array},$$

then $f'_*(\mathcal{O}_{X'})$ is the pull-back of $f_*(\mathcal{O}_X)$. This follows from the general formalism. Note if that f satisfies these conditions, then so does the composite—in either direction—of f with any finite étale morphism. One may see this by reducing to the case where Y is complete local(!) by flat base change, and then using the above description of the functor T.

Finally, we are able to construct the not-so-long exact sequence of a "fibration."

Theorem 3.9 (Exact sequence)**.** *Let $f : X \to Y$ be a proper, flat morphism of connected noetherian schemes with geometrically reduced fibers. Suppose $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$, and let $\overline{y} \in Y$ be a geometric point, lifting to a point $\overline{x} \in X$. Then there is an exact sequence*

$$\pi_1(X_{\overline{y},}, \overline{x}) \to \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y}) \to 1.$$

Note that by one form of Zariski's Main Theorem, the fibers of f are geometrically connected because $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$.

To make sense of this result, we will have to think in terms of étale covers: after all, it is only in this sense that we even know what the morphisms of groups are. Namely, from the sequence of schemes $X_{\overline{y}} \to X \to Y$, we have functors $\mathrm{Et}(Y) \to \mathrm{Et}(X) \to \mathrm{Et}(X_{\overline{y}})$. If we let the three fundamental groups be $G_{X_{\overline{y}}}, G_X, G_Y$, then we have a collection of functors

$$G_Y\text{–}\mathbf{Set} \to G_X\text{–}\mathbf{Set} \to G_{X_{\overline{y}}}\text{–}\mathbf{Set}.$$

Here G–**Set** denotes the categories of finite continuous G-sets, for G a profinite group.

To say that $\varphi : G_X \to G_Y$ is surjective is to say that any connected G_Y-set, when considered as a G_X-set via φ, is also connected. Via the Galois correspondence, this means that the base-change to X of a connected étale cover of Y is a connected étale cover of X. This is easy to see.

Next, to say that the kernel of $G_X \to G_Y$ is contained in the image of $G_{X_{\overline{y}}}$ is to say that if a connected G_X-set (say, of the form G_X/H for H an open subgroup) is such that its restriction to a $G_{X_{\overline{y}}}$-set is trivial (i.e. H contains the image of $G_{X_{\overline{y}}}$), then G_X arises as a G_Y-set.

That is, if $X' \to X$ is a connected étale cover such that $X' \times_X X_{\overline{y}}$ is trivial, then X' is the base-change of an étale cover of Y.

Proof. We shall first start by establishing the surjectivity of $\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$. Let $Y' \to Y$ be a connected étale cover. We want to show that $X \times_Y Y'$ is connected; this is equivalent to surjectivity, by the above discussion. But $f' : X \times_Y Y' \to Y'$ satisfies the same conditions as f (because push-forward commutes with flat base change!). Now f' is proper and surjective, so it is a quotient map. Also, its fibers are connected by Zariski's Main Theorem. It follows formally that $X \times_Y Y'$ is connected if Y' is.

Next, it is clear that the composite $\pi_1(X_{\overline{y}}, \overline{x}) \to \pi_1(Y, \overline{y})$ is trivial, because the map $X_{\overline{y}} \to Y$ itself is a constant map.

The hardest part will be to show exactness at the middle. As we discussed above, this amounts to saying that if $X' \to X$ is a connected étale cover such that $X' \times_X X_{\overline{y}} = X' \times_Y \overline{y}$ is trivial, then X' is the base-change of an étale cover of Y. Here, we consider the Stein factorization of $g : X' \to Y$; we get a composite $X' \to Y' = \mathbf{Spec} g_*(\mathcal{O}_{X'}) \to Y$, where as we have seen, $Y' \to Y$ is an étale cover. The claim is that the diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is cartesian. This will imply what we want.

To see this, note that $X \times_Y Y' \to X$ is an étale cover, and there is a natural map $X' \to X \times_Y Y'$. This is necessarily an étale cover itself, and it is in fact surjective: this follows because $X' \to Y'$ is surjective (as $X' \to Y$ is surjective and $Y' \to Y$ is finite), so we need to see that $X' \to X \times_Y Y'$ has degree one.

To check this, however, we can base-change to $X_{\overline{y}}$ (as both schemes are connected, we just need to check the degree at one point). This is the same as base-changing both X' and $X \times_Y Y'$ to $\overline{y}$ over Y. To do this, we will show that

$$X' \times_Y \overline{y} \to X \times_Y Y' \times_Y \overline{y} \tag{2}$$

is an isomorphism. To do this, note first that since $g : X' \to Y$ is proper and flat with geometrically reduced fibers as above, the formation of $g_*(\mathcal{O}_X)$ commutes with arbitrary base-change. Moreover, these properties are preserved under base change themselves. The upshot is that we can just assume Y is the spectrum of an algebraically closed field.

Now X is connected, and X' splits as a sum of copies of X, so it is clear (because X is reduced) that Y' is a sum of copies of the appropriate number of copies of k, and that the map $X' \to X \times_Y Y'$ is an isomorphism. □

4. The fundamental group of a smooth projective curve

4.1. **The fundamental group of a product.** We now want to show that étale π_1 behaves nicely with respect to products, at least when one factor is proper. We first, however, need a result that states that calculating the π_1 of a variety in characteristic zero is equivalent to calculating it over $\mathbb{C}$. To prove this special case, we shall "bootstrap" by proving $\pi_1(X \times_k Y) = \pi_1(X) \times \pi_1(Y)$ when both are *varieties* over k (i.e. admit k-points). Then we shall prove this (by a projective limit argument) when Y is the spectrum of a field, which is the result that changing the algebraically closed base field has no effect on the fundamental group (at least for a proper scheme).

Lemma 4.1. *Let $X \to \mathrm{Spec} k$ be a proper, integral variety over an algebraically closed field k. Let $Y \to \mathrm{Spec} k$ be a noetherian scheme. Let $\overline{z} \to X \times_k Y$ be a geometric point, and let $\overline{x} \to X, \overline{y} \to Y$ be the composites to X, Y. Suppose $\overline{z}$ is itself isomorphic to* Spec$k$ *and the various maps are maps over k. Then*

$$\pi_1(X \times_k Y, \overline{z}) \simeq \pi_1(X, \overline{x}) \times \pi_1(Y, \overline{y})$$

under the natural projection maps.

In the theory of the topological fundamental group, this result is immediate, but it is harder for the étale fundamental group.

Proof. We consider the map

$$\pi : X \times_k Y \to Y,$$

which is proper and flat, with geometrically reduced fibers. One deduces from the Künneth formula that $\pi_*(\mathcal{O}_{X \times_k Y}) = \mathcal{O}_Y$. As a result, we get a short exact sequence

$$\pi_1(X \times_k \overline{y}, \overline{y}) \to \pi_1(X \times_k Y, \overline{z}) \to \pi_1(Y, \overline{y}) \to 1.$$

The first term is $\pi_1(X, \overline{x})$, though, because the geometric fiber of $X \times_k Y \to Y$ under $\mathrm{Spec} k = \overline{y} \to Y$ is just X.

Now the result is "formal." The natural map given above $\pi_1(X, \overline{x}) \to \pi_1(X \times_k Y, \overline{z})$ has a section (namely, the projection). It follows that there is a split exact sequence

$$1 \to \pi_1(X, \overline{x}) \to \pi_1(X \times_k Y, \overline{z}) \to \pi_1(Y, \overline{y}) \to 1,$$

with the first map admitting a section. It follows from this that the natural map $\pi_1(X \times_k Y, \overline{z}) \to \pi_1(X, \overline{x}) \times \pi_1(Y, \overline{y})$ must be an isomorphism. For, the above diagram shows it is injective, and surjectivity follows because we can construct a section by the inclusions $X \to X \times_k Y, Y \to X \times_k Y$ using the geometric points as above.

Let us make an important remark here: the only place we used that $\overline{z} = \mathrm{Spec} k$ was to argue that the geometric fiber of $X \times_k Y$ had the same fundamental group as that of X. In general, the geometric fiber would be a base-change of X to some other algebraically closed field. If we show that the algebraically closed base is irrelevant, then the above lemma will be seen to be true without any hypothesis on $\overline{z}$. □

It follows now that if X, Y are finite type over an algebraically closed field k and connected, and X is proper over Spec$k$, then the above result holds for *any* geometric point $\overline{z} \to X \times_k Y$, because the fundamental group is independent of the geometric point chosen, and because we can always find geometric points $\mathrm{Spec}k \to X, Y$ as they are "varieties."

We now want to apply the general formalism of EGA IV-8 of "projective limits of schemes" to the étale fundamental group.

Lemma 4.2. *Let $\{A_\alpha\}$ be an inductive system of rings and $\{X_\alpha\}$ a projective system of schemes over the projective system $\{\mathrm{Spec}A_\alpha\}$. Suppose the maps $X_\alpha \to \mathrm{Spec}A_\alpha$ are of finite presentation, and are such that whenever $\alpha \leq \beta$, the diagram*

$$\begin{array}{ccc} X_\beta & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ \mathrm{Spec}A_\beta & \longrightarrow & \mathrm{Spec}A_\alpha \end{array}$$

is cartesian. Let $\overline{x} \to X$ be a geometric point and let $\overline{x}_\alpha \to X_\alpha$ be the images. Then the natural map

$$\pi_1(X, \overline{x}) \to \varprojlim \pi_1(X_\alpha, \overline{x}_\alpha)$$

is an isomorphism.

Proof. This follows from the fact that to give an étale cover of X is the same as giving a compatible family of étale covers of X_α for α large enough, which in turn is a general consequence of the "projective limit" formalism in EGA IV-8. □

Lemma 4.3. *Let $X \to \mathrm{Spec}k$ be a proper, integral variety over an algebraically closed field k. Let K be an extension of k which is also algebraically closed. Then, for any geometric point $\overline{x} \to X_K = X \times_k K$, the natural map*

$$\pi_1(X_K, \overline{x}) \to \pi_1(X, \overline{x})$$

is an isomorphism.

Proof. Indeed, for each finitely generated integral k-algebra contained in K, we have that

$$\pi_1(X \times_k \mathrm{Spec}A, \overline{x}) = \pi_1(X, \overline{x}) \times \pi_1(\mathrm{Spec}A, \overline{x}).$$

We now take the projective limit as A ranges over all finitely generated k-algebras contained in K. Then the latter term in the product tends to $\pi_1(\mathrm{Spec}K, \overline{x}) = 1$, so we get the result. □

As indicated in the end of the proof, this allows us to generalize theorem 4.1 by removing the hypothesis on the geometric point.

Corollary 4.4. *Let $X \to \mathrm{Spec}k$ be a proper, integral variety over an algebraically closed field k. Let $Y \to \mathrm{Spec}k$ be a noetherian scheme. Let $\overline{z} \to X \times_k Y$ be a geometric point, and let $\overline{x} \to X, \overline{y} \to Y$ be the composites to X, Y. Then*

$$\pi_1(X \times_k Y, \overline{z}) \simeq \pi_1(X, \overline{x}) \times \pi_1(Y, \overline{y})$$

under the natural projection maps.

4.2. **Completion of the proof.** Finally, we have all the main ingredients necessary to complete the proof that the étale fundamental group of a smooth projective curve of genus g admits $2g$ topological generators.

We first claim:

Lemma 4.5. *Consider a smooth projective curve $X_0 \to \mathrm{Spec} k$ where k is algebraically closed of characteristic zero. Then $\pi_1(X_0, \overline{x})$ admits $2g$ topological generators.*

Proof. Indeed, by "noetherian descent," we note that k is the colimit of the subfields $k' \subset k$ which are the algebraic closures of finitely generated extensions of $\mathbb{Q}$. As a result, there is a cartesian diagram for some k':

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0' \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} k'. \end{array}$$

Here, by choosing k' appropriately, noetherian descent allows us to assume that $X_0' \to \mathrm{Spec} k'$ is itself a smooth projective curve. We know that the fundamental group of X_0 is the same as that of X_0', so we can reduce to proving the result for X_0'. But k' embeds in $\mathbb{C}$, so we can base-change to $\mathrm{Spec}\mathbb{C}$. Then, however, the result is a consequence of the Riemann existence theorem and the ordinary theory of the topological fundamental group for a compact topological surface. $\square$

It is now clear how to finish the proof of Grothendieck's theorem, following the "scheme" sketched above. Namely, let $X_0 \to \mathrm{Spec} k$ be a smooth projective curve over a field of characteristic p. Choose a smooth, proper lifting $X \to \mathrm{Spec} A$ for A a complete DVR of unequal characteristic. We know that $\pi_1(X) = \pi_1(X_0)$ (where we have omitted the geometric points for simplicity), so we are reduced to showing that $\pi_1(X)$ is generated appropriately.

Then $X \to \mathrm{Spec} A$ is flat and has geometrically reduced fibers (as a smooth morphism). Moreover, $f_*(\mathcal{O}_X) = \mathcal{O}_{\mathrm{Spec} A}$. To see this, note that $\Gamma(X, \mathcal{O}_X) = B$ is a finitely generated A-module, and it is finite étale in view of theorem 3.5; it is thus a product of copies of A, which is complete. Since X is integral, as a smooth scheme over a DVR, it follows that $\Gamma(X, \mathcal{O}_X) = A$.

Let $\overline{\xi}$ be a geometric point of $\mathrm{Spec} A$ mapping to the generic point. It follows that the exact sequence applies and the map

$$\pi_1(X_{\overline{\xi}}) \to \pi_1(X)$$

is surjective, because $\mathrm{Spec} A$ has no nontrivial étale covers. We have already seen that $\pi_1(X_{\overline{\xi}})$ has $2g$ topological generators; indeed, note that the genus of $X_{\overline{\xi}}$ is that of X_0, because the genus is constant in flat families by the semicontinuity theorem. It follows that $\pi_1(X) = \pi_1(X_0)$ has $2g$ topological generators.

References

1. Alexander Grothendieck, *SGA 1: Revêtements étales et groupe fondamental*, Springer, 1971, arXiv:math/0206203v2.
2. Alexander Grothendieck and Jean Dieudonné, *Élements de géometrie algébrique*, Publications Mathématiques de l'IHÉS.
3. J.P. Murre, *Lectures on Grothendieck's theory of the fundamental group*, Tata Institute of Fundamental Research, 1967.
4. Jean-Pierre Serre, *Local fields*, Springer-Verlag, 1979.