

# NOTES ON CARTIER AND WEIL DIVISORS

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ABSTRACT. These are notes on divisors from Ravi Vakil's book [2] on scheme theory that I prepared for the Foundations of Algebraic Geometry seminar at Harvard. Most of it is a rewrite of chapter 15 in Vakil's book, and the originality of these notes lies in the mistakes. I learned some of this from [1] though.

Recall:

**Definition 0.1.** A **line bundle** on a ringed space  $X$  (e.g. a scheme) is a locally free sheaf of rank one. The group of isomorphism classes of line bundles is called the **Picard group** and is denoted  $\text{Pic}(X)$ .

Here is a standard source of line bundles.

## 1. THE TWISTING SHEAF

1.1. **Twisting in general.** Let  $R$  be a graded ring,  $R = R_0 \oplus R_1 \oplus \dots$ . We have discussed the construction of the scheme  $\text{Proj}R$ . Let us now briefly explain the following additional construction (which will be covered in more detail tomorrow). Let  $M = \bigoplus M_n$  be a graded  $R$ -module.

**Definition 1.1.** We define the sheaf  $\widetilde{M}$  on  $\text{Proj}R$  as follows. On the basic open set  $D(f) = \text{Spec}R_{(f)} \subset \text{Proj}R$ , we consider the sheaf associated to the  $R_{(f)}$ -module  $M_{(f)}$ . It can be checked easily that these sheaves glue on  $D(f) \cap D(g) = D(fg)$  and become a quasi-coherent sheaf  $\widetilde{M}$  on  $\text{Proj}R$ .

Clearly, the association  $M \rightarrow \widetilde{M}$  is a functor from graded  $R$ -modules to quasi-coherent sheaves on  $\text{Proj}R$ . (For  $R$  reasonable, it is in fact *essentially* an equivalence, though we shall not need this.)

We now set a bit of notation.

**Definition 1.2.** If  $M$  is a graded  $R$ -module, we let  $M(k)$  be the graded  $R$ -module defined via  $M(k)_n = M_{n+k}$ , with the  $R$ -multiplication defined in the obvious manner. (This is isomorphic to  $M$  as an  $R$ -module, but generally not as a graded  $R$ -module.)

**Definition 1.3.** Let  $R$  be a graded ring as above. The sheaf  $\widetilde{R(k)}$  on  $\text{Proj}R$  associated to the graded  $R$ -module  $R(k)$  is denoted  $\mathcal{O}(k)$ . The sheaf  $\mathcal{O}(1)$  is called the **twisting sheaf** of Serre.

**Example.**  $\mathcal{O}(0)$  is the structure sheaf.

This is really most interesting only when the ring  $R$  satisfies a reasonable condition.

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**Proposition 1.4.** *If  $R$  is generated by  $R_1$  as  $R_0$ -algebra, then the  $\mathcal{O}(k)$  are free. Further,  $\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \simeq \mathcal{O}(k_1 + k_2)$  for any  $k_1, k_2 \in \mathbb{Z}$ .*

*Proof.* Indeed, let  $f \in R_1$ . Then we have an isomorphism of  $R_{(f)}$ -modules

$$R(k)_{(f)} \simeq R_{(f)}$$

given by dividing by  $f^k$ . So  $\widetilde{R(k)} = \mathcal{O}(k)$  is free on each  $\text{Spec}R_{(f)}$ . But these cover  $\text{Proj}R$  as  $R$  is generated by  $R_1$ .

The second part follows from the next result applied to  $M = R(k_1), N = R(k_2)$ :

**Proposition 1.5.** *Let  $R$  be a graded ring generated by  $R_1$  as  $R_0$ -algebra. Then if  $M, N$  are graded  $R$ -modules, the canonical homomorphism*

$$\widetilde{M} \otimes \widetilde{N} \rightarrow \widetilde{M \otimes N}$$

*is bijective.*

▲

**1.2. The case of projective space.** It is useful to look at this twisting procedure in case  $R = A[x_0, \dots, x_n]$  for  $A$  a ring, in which case  $\text{Proj}R = \mathbb{P}_A^n$ . In this case, the elements of degree one generate the ring, and the twisting sheaves  $\mathcal{O}(k)$  are line bundles.

We can in fact compute their sections.

**Proposition 1.6.** *Let  $X = \mathbb{P}_A^n$  for a ring  $A$ . Then  $\Gamma(X, \mathcal{O}(k))$  is the space of homogeneous polynomials in  $x_0, \dots, x_n$  of degree  $k$ . (In particular, it is zero for  $k < 0$ .)*

*Proof.* We begin with a general observation. Let  $M$  be a graded  $R$ -module over some graded ring  $R$ . There is a natural (functorial) map

$$M_0 \rightarrow \Gamma(\text{Proj}R, \widetilde{M})$$

because  $M_0$  maps into  $M_{(f)}$  for any  $f \in R$  in a natural way. **In general, this is not an isomorphism.**

Here  $M = R(k)$ ,  $R = A[x_0, \dots, x_n]$ . The elements of degree zero are the elements of degree  $k$  in  $R$ , i.e. the homogeneous polynomials of degree  $k$ . So this is the natural isomorphism, but we should check that it is in fact an isomorphism.

We can do this as follows.  $\mathbb{P}_A^n$  is covered by open sets  $D(x_i), 0 \leq i \leq n$ . Over  $D(x_i)$ , we have that the global sections are rational functions with  $x_i$ 's in the denominator and of homogeneous degree  $k$ . But the intersection of any two  $A[x_1, \dots, x_n]_{x_i}$  inside the full ring  $A[x_1, \dots, x_n]_{x_1 \dots x_n}$  is just the polynomial ring. This implies the result.

▲

**Corollary 1.7.** *If  $m \neq m'$ , then  $\mathcal{O}(m) \not\cong \mathcal{O}(m')$  on  $\mathbb{P}_A^n$ . In particular, there is an injection of groups*

$$\mathbb{Z} \rightarrow \text{Pic}(\mathbb{P}_A^n).$$

*Proof.* If one of  $m, m'$  is nonnegative, this is clear by taking the module of global sections. If both are negative, dualization sends  $\mathcal{O}(m)$  to  $\mathcal{O}(-m)$ , so we are done in this case too.

▲

**Example.** Sheafification is necessary when tensoring sheaves. Consider  $\mathcal{F} = \mathcal{O}(1), \mathcal{G} = \mathcal{O}(-1)$  on  $X = \mathbb{P}_A^n$ . Then  $\mathcal{G}(X) = 0$ , so  $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} \mathcal{G}(X) = 0$ , but  $(\mathcal{F} \otimes \mathcal{G})(X) = A \neq 0$ .

**Example** (Essential exercise, 15.1C in Vakil).  $\dim_A \Gamma(\mathbb{P}_A^n, \mathcal{O}(m)) = \binom{n+m}{n}$ . This follows from some easy combinatorics. Indeed, this is the space of homogeneous polynomials in  $n+1$  variables of degree  $m$ . If one has  $m+n$  slots, and chooses  $n$  of them to be separators, then one can get a unique monomial of degree  $m$  in  $n+1$  variables. These form a basis for  $\Gamma(\mathbb{P}_A^n, \mathcal{O}(m))$ , so the claim is proved.

Finally, we discuss the explicit construction via transition functions of  $\mathcal{O}(m)$ .

**Proposition 1.8.** *For  $X = \mathbb{P}_A^n$ ,  $\mathcal{O}(m)$  can be described as follows. The sheaf is free on  $D(x_i), 0 \leq i \leq n$ . The transition functions between  $D(x_i)$  and  $D(x_j)$  are given by  $\frac{x_i^m}{x_j^m}$  (which is a regular function on  $D(x_i) \cap D(x_j)$ ).*

This explicit description makes it clear that  $\mathcal{O}(m_1) \otimes \mathcal{O}(m_2) = \mathcal{O}(m_1 + m_2)$ , since tensoring line bundles corresponds to multiplying the changes-of-coordinates.

*Proof.* Follows easily from the explicit construction of the local freeness. The map  $\mathcal{O}(m)|_{D(x_i)} \rightarrow \mathcal{O}|_{D(x_i)}$  was given by division by  $x_i^m$ . The map  $\mathcal{O}(m)|_{D(x_j)} \rightarrow \mathcal{O}|_{D(x_j)}$  was division by  $x_j^m$ .  $\blacktriangle$

Clearly, these transition functions satisfy the cocycle condition.

**Example.** Let us now consider the case of **complex** projective space  $\mathbb{C}\mathbb{P}^n$ , and look at topological (as opposed to simply algebraic) line bundles, and see what the  $\mathcal{O}(m)$  look like.

The **tautological line bundle** on  $\mathbb{C}\mathbb{P}^n$  (corresponding to pairs  $(\ell, x)$  where  $\ell \in \mathbb{C}\mathbb{P}^n$  and  $x \in \mathbb{C}^{n+1}$  with  $x \in \ell$ ) corresponds to  $\mathcal{O}(-1)$ . Indeed, on each open set  $D(x_i)$  we can trivialize this bundle by sending a number  $t \in \mathbb{C}$  and a line  $\ell = [\ell_0 : \dots : \ell_n] \in \mathbb{C}\mathbb{P}^n$  to  $x = t(\ell_0/\ell_i, \dots, \ell_n/\ell_i) \in \mathbb{C}^{n+1}$ .

So given a line  $\ell = [\ell_0 : \dots : \ell_n]$  and a point  $x = (x_0, \dots, x_n)$ , we get a number from this by finding  $t$  such that

$$x = t(\ell_0/\ell_i, \dots, \ell_n/\ell_i)$$

or equivalently,

$$t = x_i.$$

This is the trivialization on the set  $D(x_i)$ . To from  $D(x_i)$  to  $D(x_j)$ , one has to go from  $x_i$  to  $x_j$ , so the multiplication factor is

$$x_j/x_i = \ell_j/\ell_i$$

which is the inverse of what it is for  $\mathcal{O}(1)$ .

## 2. WEIL DIVISORS

**2.1. First definition.** Let  $X$  be a noetherian scheme.

**Definition 2.1.** A **Weil divisor** on  $X$  is a formal integer combination of integral codimension one closed subvarieties of  $X$ . So the group of Weil divisors is free abelian on the codimension one closed subvarieties.

A Weil divisor is called **effective** if it is a nonnegative linear combination of subvarieties. The **support** of a Weil divisor is the union of the subvarieties that occur.

Suppose now that  $X$  is integral and regular in codimension one. The latter means that for any codimension one point  $x$ , the local ring  $\mathcal{O}_x$  is a discrete valuation ring. Let  $f \in k(X)$  be a nonzero rational function, i.e. a section defined over a dense open subset of  $X$ .

**Definition 2.2.** If  $y \in X$  is a codimension one point, then there is a discrete valuation on  $\mathcal{O}_y$ , and thus on the quotient field  $k(X)$ . The valuation of  $f$  is denoted  $v_y(f)$  or  $v_Y(f)$  if  $y$  is the generic point of the subscheme  $Y$ .

**Remark.** This can also be defined for a reduced (not necessarily irreducible) scheme by splitting into irreducible components.

We remark that the divisors can be restricted to open sets.

**Proposition 2.3.**  $v_y(f) = 0$  for almost all codimension one points  $y \in Y$ .

*Proof.* Assume  $U \subset X$  is an open set on which  $f$  is a unit. Then  $v_y(f) = 0$  if  $y \in U$ . But  $X - U$  is a finite union of subvarieties of proper codimension. In particular, there are only finitely many codimension one points in  $X - U$ . This proves the result.  $\blacktriangle$

In view of this, we can make:

**Definition 2.4.** Given  $f \in k(X)^*$ , we define the **divisor of  $f$** , denoted  $\text{div}(f)$ , to be

$$\sum_{y \text{ codim } 1} v_y(f)(Y),$$

where  $y$  is the generic point of  $Y$ .

It is clear from the properties of discrete valuations that  $\text{div}(fg) = \text{div}(f) + \text{div}(g)$ . In particular, we can define a subgroup of the Weil divisors consisting of the principal divisors. The quotient group is called the **class group** of  $X$ .

**Definition 2.5.** We write  $\text{Cl}(X)$  for the (Weil) divisor class group.

**2.2. The divisor of a line bundle section.** We would like to generalize this to line bundles. Let  $\mathcal{L}$  be a line bundle on  $X$  and  $s$  a rational section (i.e. a section defined over a dense open set).

**Definition 2.6.** We define  $\text{div}(s)$  as follows. Given a closed integral subscheme  $Y$  of codimension one, with generic point  $y$ , we choose a neighborhood  $U$  of  $y$  on which  $\mathcal{L}$  is trivial. On this neighborhood,  $s$  corresponds to a rational function  $f$  on  $U$  (the choice of rational function is determined up to an element of  $\mathcal{O}(U)^*$ ). We define  $v_Y(s) = v_Y(f)$ . It is clear that this does not depend on the trivialization.

Clearly  $\text{div}$  does not depend on multiplication by global units. This also induces a map  $\text{div} : \{\text{rational sections of } \mathcal{L}\} / \{\text{action of } \mathcal{O}(X)^*\} \rightarrow \text{Weil}(X)$ .

**Proposition 2.7.** *If  $X$  is normal,  $\text{div}$  is injective. Alternatively, a rational section whose divisor is zero is everywhere defined and invertible.*

*Proof.* This is local, so we can assume that  $\mathcal{L} = \mathcal{O}_X$  and  $X$  affine. Then we have a rational function regular in codimension one, so the algebraic Hartogs lemma (by normality) implies the result.  $\blacktriangle$

**Example** (15.2A in Vakil). Let  $k$  be a field. Let us compute  $\operatorname{div}(x^3/(x+1))$  on  $\mathbb{A}_k^1$ .<sup>1</sup> Here the only pole is at  $x = -1$  (that is, the prime ideal  $(x+1)$ ), and the valuation there is  $-1$ . The zero is at  $x = 0$  (that is, the prime ideal  $(x)$ ) and the zero there is of order 3. In particular, the divisor is  $3[0] - [-1]$ .

**Example** (15.2A in Vakil, contd.). Consider  $\mathcal{O}(1)$  on  $\mathbb{P}_k^1$  and the rational section  $X^2/(X+Y)$ . Let us compute the divisor.

There are two basic open sets  $D(X)$  and  $D(Y)$ . On the former, the trivialization is by dividing by  $X$ , so we get  $X/(X+Y) = 1/(1+(X/Y))$ . This has a simple pole when  $X/Y = -1$ . So at the point  $[-1 : 1]$ , there is a simple pole of this divisor.

The second trivialization is by dividing by  $Y$ , in which case we get  $X^2/((X+Y)Y)$ , which has a zero of order 2 at  $X = 0$ . So the total divisor is

$$2[0 : 1] - [-1 : 1].$$

### 3. THE CORRESPONDENCE WITH CARTIER DIVISORS

**3.1. The sheaves  $\mathcal{L}(D)$ .** Fix a noetherian integral scheme  $X$ , whose codimension one points are regular. Assume furthermore that  $X$  is *normal*.

We shall now describe a construction that associates to any Weil divisor  $D$  a quasi-coherent sheaf  $\mathcal{L}(D)$ , which under smoothness hypotheses (or local principality of  $D$ ) will be invertible.

**Definition 3.1.** Let  $D$  be a Weil divisor. For every open set  $U$ , consider the set of  $f \in k(X)$  such that  $(\operatorname{div} f + D)_U \geq 0$ . This is a sheaf of  $\mathcal{O}_X$ -modules, denoted  $\mathcal{L}(D)$ .

Outside the support of  $D$ , this is the same thing as the structure sheaf. (A function which is regular in codimension one on an open set is regular on that open set by algebraic Hartogs.)

**Proposition 3.2.**  $\mathcal{L}(D)$  is quasi-coherent.

*Proof.* We shall show that on affines, we have  $\Gamma(\operatorname{Spec} A, \mathcal{L}(D))_f = \Gamma(\operatorname{Spec} A_f, \mathcal{L}(D))$ . This suffices for quasi-coherence. There is always a map in one direction. So suppose  $s \in \Gamma(\operatorname{Spec} A_f, \mathcal{L}(D))$ . There is an open set  $U \subset \operatorname{Spec} A_f$  on which  $s$  is regular, and we know that

$$(\operatorname{div} s + D)_{\operatorname{Spec} A_f} \geq 0.$$

So we know that  $\operatorname{div}(s)|_{\operatorname{Spec} A}$  can fail to satisfy  $(\operatorname{div} s + D) \geq 0$  only at points in  $V((f)) \subset \operatorname{Spec} A$ . But if we multiply  $s$  by a high power of  $f$ , it will raise the divisors at codimension one points in  $V((f))$  and at worst do nothing elsewhere. We find that for large  $N$ ,

$$(\operatorname{div} f^N s + D)_{\operatorname{Spec} A} \geq 0.$$

This is precisely the claim in question. ▲

In general, we can say no more. But oftentimes, under smoothness hypotheses, we will have that  $\mathcal{L}(D)$  is actually an invertible sheaf.

Let  $D$  be a divisor. We say that  $D$  is **locally principal** if for each  $x \in X$ , there is a neighborhood  $U$  of  $x$  on which  $D|_U$  is principal.

<sup>1</sup>This is clearly integral, and is regular as the localizations of a polynomial ring are regular local rings.

**Proposition 3.3.** *Let  $X$  be a normal scheme. If  $D$  is locally principal, then  $\mathcal{L}(D)$  is an invertible sheaf.*

*Proof.* Let  $x \in X$ . There is a neighborhood  $U$  of  $x$  on which  $D|_U$  is described by a rational function  $f \in k(U)$ . Then there is an isomorphism

$$\mathcal{O}_X|_U \simeq \mathcal{O}(D)|_U$$

sending  $g \rightarrow g/f$ . The fact that it is an isomorphism follows from the algebraic Hartogs lemma.  $\blacktriangle$

An important case when this holds is:

**Proposition 3.4.** *Let  $X$  be factorial (in addition to noetherian, normal, and integral). Then any divisor on  $X$  is locally principal.*

Since regular local rings are factorial, smooth schemes are OK here.

*Proof.* Let  $D$  be a divisor on  $X$ . Without loss of generality,  $D$  itself corresponds to a single closed subscheme. Let  $x \in X$ . Then  $\mathcal{O}_x$  is a UFD.  $D$  corresponds to an ideal of height one in  $\mathcal{O}_x$ , which is principal by factoriality. So there is a generator of the ideal of  $D$  in  $\mathcal{O}_x$ . This generator must have the same divisor as  $D$  in some small neighborhood.  $\blacktriangle$

In particular, on a smooth scheme,  $\mathcal{L}(D)$  is *always invertible*.

**3.2. The map  $\text{Pic}(X) \rightarrow \text{Cl}(X)$ .** Next, we claim that the map as above is a type of half-inverse, in a certain sense.

**Proposition 3.5.** *Let  $X$  be normal. Let  $\mathcal{L}$  be an invertible sheaf and  $s$  a rational section. Then  $\mathcal{O}(\text{div}s)$  is invertible and in fact there is an isomorphism  $\mathcal{O}(\text{div}s) \rightarrow \mathcal{L}$ . In particular, this induces a morphism of groups*

$$\text{Pic}(X) \rightarrow \text{Cl}(X)$$

*Proof.* Let  $f \in \Gamma(U, \mathcal{O}(\text{div}s))$ . Then map  $f \rightarrow fs \in \Gamma(U, \mathcal{L})$ . This will have no poles, so by normality is regular.

Given  $\mathcal{L}$ , we choose a rational section  $s$  and send it to  $\text{div}s \in \text{Cl}(X)$ . Any two rational sections differ by a rational function so we get a map  $\text{Pic}(X) \rightarrow \text{Cl}(X)$ . It is easy to check that this is additive.  $\blacktriangle$

Note that this map is defined on any (noetherian normal) scheme.

**3.3. The reverse map.** Let now  $X$  be a *factorial* (e.g. smooth) scheme, in addition to noetherian and integral.<sup>2</sup> Then we have a map

$$\text{Cl}(X) \rightarrow \text{Pic}(X)$$

sending a divisor  $D$  to its associated  $\mathcal{L}(D)$ , which we have seen it is invertible. It is easy to check that this is a homomorphism, i.e.  $\mathcal{L}(D_1 + D_2) \simeq \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$ .

**Theorem 3.6.** *On a factorial noetherian integral scheme,  $\text{Cl}(X) \simeq \text{Pic}(X)$  under these isomorphisms.*

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<sup>2</sup>Normality is implied by factoriality.

*Proof.* Given a line bundle  $\mathcal{L}$ , we have seen that we can realize  $\mathcal{L}$  as  $\mathcal{O}(\text{div}s)$  for  $s$  a rational section. So the map  $\text{Cl}(X) \rightarrow \text{Pic}(X)$  is surjective.

It is also injective. If  $\mathcal{O}(D)$  is principal, with generating section  $f$ , then  $f$  generates the stalks  $\mathcal{O}(D)_x$  at any codimension one point  $x$ . These stalks are  $\mathcal{O}_x$  if  $x \notin \text{supp}D$ . If  $x \in D$  with multiplicity  $n \in \mathbb{Z}$ , then  $f$  must be an  $-n$ th power of a uniformizer. It follows that  $f^{-1}$  cuts out the divisor  $D$ . Whence  $D$  is principal.

Let us also check that the two constructions  $D \rightarrow \mathcal{L}(D)$ ,  $\mathcal{L} \rightarrow \text{div}s$  are inverses. If  $\mathcal{L}$  is an invertible sheaf, we have seen that  $\mathcal{L}(\text{div}s) \simeq \mathcal{L}$ . If  $D$  is a divisor, it follows from the above that  $D$  must be equivalent to the divisor of  $\mathcal{L}(D)$ .

We can check this directly (and perhaps more effectively). There is a rational section 1 representing the function 1. Let us cover  $X$  by open sets  $U_i$  on which  $D$  is principal, say cut out by  $f_i \in k(U_i) = k(X)$ . Then the divisor of 1 as a rational section of  $\mathcal{L}(D)$  is what one gets by gluing the  $\text{div}(f_i)|_{U_i}$ , which is just  $D$ .  $\blacktriangle$

#### 4. COMPUTATIONS

We shall now discuss how to compute these groups.

**4.1. Weil divisors on open subsets.** We now prove a useful exact sequence. An amusing application of this will be *Nagata's lemma*.

**Proposition 4.1** (Exact sequence). *Let  $X$  be an integral scheme, regular in codimension one, and let  $Z \subset X$  be an integral closed subscheme of codimension one. Then there is an exact sequence*

$$\mathbb{Z} \xrightarrow{n \rightarrow n[Z]} \text{Cl}(X) \rightarrow \text{Cl}(X - Z) \rightarrow 0.$$

Here the second map is restriction.

*Proof.* Clearly the composite is zero, since  $Z$  restricts to the trivial divisor on  $\text{Cl}(X - Z)$ . The last map is surjective since given a divisor in  $X - Z$ , we can take its closure in  $X$ . (Alternatively, we can think via generic points.)

We just need to check exactness in the middle. Suppose  $D \in \text{div}(X)$  is principal on  $X - Z$ . Then, by multiplying by the rational function that makes it principal there, we can assume that it is supported in  $Z$ . Then it is a multiple of  $Z$ .  $\blacktriangle$

Finally, we note:

**Proposition 4.2.** *Let  $Z \subset X$  be of codimension at least 2. Then the canonical map*

$$\text{Cl}(X) \rightarrow \text{Cl}(X - Z)$$

*is an isomorphism.*

*Proof.* Weil divisors only see codimension one.  $\blacktriangle$

**Example.**  $\text{Cl}(\mathbb{A}^2 - \{*\}) = \text{Cl}(A^2)$ . This last object is zero, as we shall soon see, because a polynomial ring is factorial.

#### 4.2. Weil divisors on affine schemes.

**Proposition 4.3.** *A noetherian ring  $A$  (regular in codimension one) is a UFD if and only if it is normal and  $\text{Cl}(\text{Spec}A) = 0$ .*

*Proof.* If  $A$  is a UFD, then it is normal. In addition, every prime ideal of height one is principal. This means that every Weil divisor is principal.

Conversely, let  $A$  be normal and suppose  $A$  satisfies  $\text{Cl}(\text{Spec}A) = 0$ . Then each Weil divisor is principal. This means that for each prime  $\mathfrak{p}$ , there is  $x \in A$  such that  $\text{div}x = \mathfrak{p}$ . We see from this that  $x \in A$  (by the algebraic Hartogs lemma) and if  $y \in \mathfrak{p}$ , then  $y/x \in A$  (again by algebraic Hartogs). Thus

$$(x) = \mathfrak{p}.$$

▲

**Example.** The ideal class group of a Dedekind domain is trivial iff the domain admits unique factorization. In fact, the ideal class group is just the group  $\text{Cl}$ .

**Corollary 4.4.**  $\text{Cl}(\mathbb{A}_k^n) = \text{Pic}(\mathbb{A}_k^n) = 0$ .

#### 4.3. The class group of projective space.

**Example.** We shall show that  $\text{Cl}(\mathbb{P}_k^n) = \mathbb{Z}$ , generated by a hyperplane  $\{x_0 = 0\}$ . Indeed, let us first note that any two hypersurfaces of degree  $d$  are rationally equivalent. The reason is that they are defined by homogeneous polynomials of degree  $d$  and their quotient is a rational function. Similarly, if  $H_1, H_2$  are hypersurfaces of degrees  $d_1, d_2$ , and  $e_1d_1 = e_2d_2$ , then

$$e_1[H_1] - e_2[H_2] = 0.$$

It follows that the image of a hyperplane (which is of degree one) generates  $\text{Cl}(\mathbb{P}_k^n)$ . Conversely, we need to show that if  $H$  is a hyperplane, then  $n[H] = 0$  in  $\text{Cl}(\mathbb{P}_k^n)$  only if  $n = 0$ . This is easily seen by looking at a rational function. This is a quotient of homogeneous polynomials in  $k[x_0, \dots, x_n]$  of *the same degree*, so the degree of the divisor cut out must be zero.

Now, let us connect this observation to the  $\text{Pic}(\mathbb{P}_k^n)$ . In fact, we know that  $\text{Pic}(\mathbb{P}_k^n) = \text{Cl}(\mathbb{P}_k^n)$  as projective space is nonsingular, so  $\text{Pic}(\mathbb{P}_k^n) = \mathbb{Z}$ . But earlier we constructed an injection  $\mathbb{Z} \rightarrow \text{Pic}(\mathbb{P}_k^n)$  sending  $m \rightarrow \mathcal{O}(m)$ . Indeed:

**Example.** If  $H$  is a hyperplane, then  $\mathcal{L}(H) \simeq \mathcal{O}(1)$  on  $\mathbb{P}_k^n$ . Indeed, to see this, choose the section  $x_0 \in \Gamma(X, \mathcal{O}(1))$ . This divisor is globally defined and generates the sheaf except at points in  $\{x_0 = 0\}$ . In fact, a look at the definitions shows that its divisor is precisely  $\{x_0 = 0\}$ , a subscheme which is linearly equivalent to any hyperplane  $H$ . The correspondence between Weil and Cartier divisors now implies that

$$\mathcal{L}(H) \simeq \mathcal{O}(1).$$

The preceding example gives a classification of line bundles on projective space:

**Corollary 4.5.** *Any line bundle on  $\mathbb{P}_k^n$  is isomorphic to some  $\mathcal{O}(m)$ .*

Finally, let us use the exact sequence in class groups and the above computation to give an example of a torsion class group.



**Example.** Let  $Y \subset \mathbb{P}_k^n$  be a hypersurface of degree  $d$ . Then we will show that

$$\mathrm{Cl}(\mathbb{P}_k^n - Y) = \mathbb{Z}/d\mathbb{Z}.$$

Indeed, we have the exact sequence

$$\mathbb{Z} \rightarrow \mathrm{Cl}(\mathbb{P}^n) \rightarrow \mathrm{Cl}(\mathbb{P}^n - Y) \rightarrow 0$$

where the image of  $1 \in \mathbb{Z}$  is  $[Y] \in \mathrm{Cl}(\mathbb{P}^n)$ . Here  $\mathrm{Cl}(\mathbb{P}_k^n) = \mathbb{Z}$  and the generator 1 of  $\mathbb{Z}$  maps to  $d$  times the generator of  $\mathrm{Cl}(\mathbb{P}_k^n)$  by the previous example. This implies the claim.

**4.4. The cone.** Heretofore all the examples have been nonsingular. Let us now look at a singular case.

**Example.** Consider the **cone**

$$X = \mathrm{Spec}k[x, y, z]/(xy - z^2)$$

over a field of characteristic not two. Let  $Z = \{x = z = 0\}$  be the closed subscheme. The nonsmooth point is at the origin and forms a locus of codimension 2. Moreover, the cone is a normal scheme.

**Lemma 4.6.**  $Z$  is integral and of codimension one.

*Proof.* We need to show that  $k[x, y, z]/(xy - z^2, x, z)$  is a domain. But this is obviously  $k[y]$ . In addition, by looking at the transcendence degree, we see that the dimensions line up.  $\blacktriangle$

So we get a map

$$\mathbb{Z} \xrightarrow{1 \rightarrow [Z]} \mathrm{Cl}(X) \rightarrow \mathrm{Cl}(X - Z) \rightarrow 0.$$

Here  $X - Z$  is the locus where  $x \neq 0$  (if  $x = 0$ , then  $z = 0$ ). So it is equivalently

$$\mathrm{Spec}k[x, y, y^{-1}, z]/(xy - z^2)$$

We can write this as  $k[y]_y[t, u]/(t - u^2)$ , and this is obviously a domain. In fact, it is  $k[y, u]_y$ , so it is a UFD. Thus its class group is zero. We find that  $[Z]$  generates  $\mathrm{Cl}(X)$ .

We will show now that  $Z$  is **not principal**, but  $2[Z]$  is. Let us start with the latter. Indeed, let us consider the divisor of  $x$ . It can only be nonzero along  $Z$ . But I claim that its valuation on  $Z$  is 2. Indeed, the local ring is

$$(k[x, y, z]/(xy - z^2))_{(x, z)}.$$

In the local ring,  $x$  is associate to  $z^2$ , and  $z$  is a uniformizer. So there we go.

Next, we must show that  $Z$  is *not principal*. We can in fact show that it is not principal at the local ring at the origin. That is, we can show that the ideal  $(x, z)$  in  $(k[x, y, z]/(xy - z^2))_{(x, y, z)}$  is not principal. This follows from the fact that  $x$  and  $z$  do not divide each other in this ring and both are prime elements.<sup>3</sup>

In particular, we find

$$\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}.$$

**Corollary 4.7.** *The ring  $k[x, y, z]/(xy - z^2)$  is not factorial.*

<sup>3</sup>This is an algebra exercise that was on the 232a problem set. It's 4 am, so I'll leave it out.

**4.5. Products with affine and projective space.** We shall now show that taking a product with  $\mathbb{A}^1$  (i.e.  $\text{Spec}\mathbb{Z}[t]$ ) doesn't affect the class group. The product with projective space adds a copy of  $\mathbb{Z}$ .

As usual, we assume that  $X$  is noetherian, integral, and regular in codimension one.

**Proposition 4.8.** *We have  $\text{Cl}(X \times \mathbb{A}^1) = \text{Cl}(X)$ . (Implicit in this is the assertion that  $X \times \mathbb{A}^1$  is regular in codimension one.)*

*Proof.* First, let us do the former. We begin with a simple question. What are the prime ideals  $\mathfrak{p}$  of height one in  $A[T]$  for  $A$  a domain? There are two cases:

- (1)  $\mathfrak{p} \cap A$  is not of height one. Then it can contain no nonzero element, or  $\mathfrak{p}$  would contain an element  $x \in A - \{0\}$ . Clearly  $x$  is a nonzerodivisor but  $\mathfrak{p}$  is not minimal over  $(x)$  (we can find  $\mathfrak{q} \subsetneq \mathfrak{p} \cap A$  containing  $(x)$ ; then  $\mathfrak{q}A[t] \subsetneq \mathfrak{p}$ ). So  $\mathfrak{p}$  must be minimal over some element  $f \in A[T] - A$ .
- (2)  $\mathfrak{p} \cap A \neq 0$  is of height one. Then  $\mathfrak{p} \cap A$  is minimal over something  $x$  in  $A$ . Then if  $\mathfrak{q} \subset A \cap \mathfrak{p}$  is minimal over  $x$ , we have  $\mathfrak{q}A[T] \subset \mathfrak{p}$ , so  $\mathfrak{p} = \mathfrak{q}A[T]$ .

Let  $K = K(A)$  be the quotient field.

In the first case, we know that  $A[T]_{\mathfrak{p}}$  is a localization of  $K[T]$  at a prime (hence maximal) ideal. In particular, there is an element  $f \in K[T]$  (we can even take  $f \in A[T]$ ) that generates the prime ideal  $\mathfrak{q}$  after localizing at  $A - \{0\}$ .

In the second case, we know that  $A[T]_{\mathfrak{p}}$  is a localization of  $A_{\mathfrak{q}}[T]$ , where  $A_{\mathfrak{q}}$  is a DVR.

**Corollary 4.9.** *In either case,  $A[T]_{\mathfrak{p}}$  is a DVR.*

Let us now return to the land of schemes, and consider a codimension one point  $x \in X \times \mathbb{A}^1$ . The above corollary states that  $x$  is a regular point (since this is local on  $X$ ).

If the projection of  $x$  to  $\mathbb{A}^1$  is dense, then we shall call  $x$  **type one**. Otherwise, we shall call  $x$  **type two**. If we shrink  $X$  to an affine neighborhood of the point, we see that type one and type two correspond precisely to the two cases above.

Now the claim is going to be that we can make any divisor on  $X \times \mathbb{A}^1$  equivalent to a sum of type two points. It is sufficient to do this for type one points. Let  $x$  be a type one point (of codimension one). Choose an affine neighborhood  $U = \text{Spec}A \subset X$  which contains the image of  $x$ . Then  $x$  corresponds to an element  $\mathfrak{p}$  of  $\text{Spec}A[t]$  of codimension one of type 1. In particular, as we remarked above, there is  $f \in A[T]$  such that  $f$  generates  $\mathfrak{p}$  in  $K[T]$ , thus in some localization  $A_r$  for  $r \in A - \{0\}$ . The upshot of all this is that  $\text{div}(f) - [\mathfrak{p}]$  will have no piece over  $\text{Spec}A_r$ , and in particular cannot contain any type one components. So this proves the first claim.<sup>4</sup>

We can construct a map  $\text{Cl}(X) \rightarrow \text{Cl}(X \times \mathbb{A}^1)$  that sends a subscheme  $V$  to  $V \times \mathbb{A}^1$ . This corresponds to sending a prime ideal  $\mathfrak{q}$  to  $\mathfrak{q}A[t]$  as above. The image is precisely the set of divisors of type two by the above reasoning. In particular,  $\text{Cl}(X) \rightarrow \text{Cl}(X \times \mathbb{A}^1)$  is surjective. We just have to check injectivity.

It will suffice to show that if  $f \in k(X \times \mathbb{A}^1)$  produces a divisor of type two, then  $f \in k(X)$  (under the canonical pull-back injection  $k(X) \hookrightarrow k(X \times \mathbb{A}^1)$ ). We can reduce to the case of  $X$  affine. Say  $X = \text{Spec}A$ , where  $A$  is a domain with quotient

<sup>4</sup>I guess, technically, I'm using the fact that a constructible set is dense iff it contains the generic point, and implicitly Chevalley's theorem.

field  $K$ . Then the fiber over the generic point is trivial, that is  $f \in K[T]$  is a unit. So  $f \in K$ . Which is what we want. This proves the result.  $\blacktriangle$

**Proposition 4.10.**  $\text{Cl}(X \times \mathbb{P}^n) = \text{Cl}(X) \oplus \mathbb{Z}$ .

*Proof.* The above result shows that  $X \times \mathbb{P}^n$  is regular in codimension one since  $\mathbb{P}^n$  is covered by affine spaces. Let  $\mathbb{P}^{n-1} \subset \mathbb{P}^n$  be the complement of the basic open set  $D(x_0) \simeq \mathbb{A}^n$ . There is an exact sequence

$$\mathbb{Z} \xrightarrow{1 \rightarrow [\mathbb{P}^{n-1} \times X]} \text{Cl}(X \times \mathbb{P}^n) \rightarrow \text{Cl}(X \times \mathbb{A}^n) \rightarrow 0.$$

The final epimorphism splits (for a divisor, take its closure). Moreover, the previous result iterated implies that  $\text{Cl}(X \times \mathbb{A}^n) = \text{Cl}(X)$ . So we just need to show that the first map is an injection. Indeed, the inverse is given by taking the degree over the generic point. (A rational function on  $X \times \mathbb{P}^n$  restricts to a rational function on  $\{\xi\} \times \mathbb{P}^n$  where  $\xi$  is the generic point—this is isomorphic to  $\mathbb{P}_{k(X)}^n$ .)  $\blacktriangle$

**Example (15.20).** Irreducible projective smooth surfaces can be birational without being isomorphic. (This is false for curves.) For instance, we have just seen that the class groups of  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$  are different (i.e. are  $\mathbb{Z}$  and  $\mathbb{Z} \oplus \mathbb{Z}$ ).

**Example.** Let  $X = \text{Spec} k[x, y, z]/(x^2 + y^2 - z^2)$  where  $\sqrt{-1} \notin k$ . We will show that  $\text{Cl}(X) = 0$  and thus that the ring is factorial. Except that **I need to add this**.

**4.6. Nagata's lemma.** We finish with a fun application of the exact sequence of Weil divisors to a purely algebraic statement about factoriality.

**Theorem 4.11.** *Let  $A$  be a noetherian domain,  $x \in A - \{0\}$ . Suppose  $(x)$  is prime and  $A_x$  is factorial. Then  $A$  is factorial.*

*Proof.* We first show that  $A$  is normal (hence regular in codimension one). Indeed,  $A_x$  is normal. So if  $t \in K(A)$  is integral over  $A$ , it lies in  $A_x$ . So we need to check that if  $a/x^n \in A_x$  is integral over  $A$  and  $x \nmid a$ , then  $n = 0$ . Suppose we had an equation

$$(a/x^n)^N + b_1(a/x^n)^{N-1} + \dots + b_N = 0.$$

Multiplying both sides by  $x^{nN}$  gives that

$$a^N \in xR,$$

so  $x \mid a$  by primality.

Now we use the exact sequence

$$(x) \rightarrow \text{Cl}(A) \rightarrow \text{Cl}(A_x) \rightarrow 0.$$

The end is zero, and the image of the first map is zero. So  $\text{Cl}(A) = 0$ . Thus  $A$  is a UFD.  $\blacktriangle$

**Example.** Let  $k$  be algebraically closed of characteristic  $\neq 2$ . Suppose  $m \geq 3$  and

$$A = k[a, b, x_1, \dots, x_n]/(ab - x_1^2 - \dots - x_m^2).$$

Then  $A$  is factorial.

First,  $A$  is a domain. We will apply Nagata's lemma with the element  $a$ . When we invert  $a$ , the ring becomes

$$k[a, a^{-1}][x_1, \dots, x_n, t]/(t - x_1^2 - \dots - x_m^2) = k[a, a^{-1}][x_1, \dots, x_m],$$

which is factorial.

We now need to show that  $(a)$  is prime. In other words, when we quotient by  $(a)$ , then we need to get a domain. But when we quotient by  $a$ , we get

$$k[b, x_1, \dots, x_n]/(x_1^2 + \dots + x_m^2).$$

Since  $m \geq 3$ , this is a domain (the polynomial  $x_1^2 + \dots + x_m^2$  is irreducible).

It is interesting to note that when  $m = 2$ , this argument fails, and indeed we have something isomorphic to a cone, which does not have a trivial class group.

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