DESCENT THEORY AND APPLICATIONS TO RING SPECTRA

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The theory of *structured ring spectra* plays an important role in modern algebraic topology. For instance, many fundamental objects in stable homotopy theory, such as real and complex K-theory and the numerous variants of topological modular forms, are $\mathbb{E}_\infty$-ring spectra and are most naturally studied as such.

Principle. To study a discrete commutative ring, we should study the objects that it acts on: that is, *modules.* Similarly, we should study ring spectra by understanding properties or features of their modules.

In particular, one of our goals will be to study invariants of modules or module categories of ring spectra.

Let R be an $\mathbb{E}_\infty$-ring spectrum. Then one has a symmetric monoidal ∞-category of R-modules. The spectrum R has homotopy groups $\pi_*(R)$ which form a graded commutative ring. Given an R-module M, the homotopy groups $\pi_*(M)$ form a graded module over $\pi_*(R)$. This is the first and most basic structure one obtains from an R-module. The theory of R-modules contains a number of homotopy coherences that one cannot detect only from as coarse an invariant as $\pi_*(M)$. Ultimately, to work with R-modules, one can use the spectral sequence

$$\mathrm{Ext}^{s,t}_{\pi_*(R)}(\pi_*(M), \pi_*(N)) \implies \pi_{t-s}\mathrm{Hom}_R(M, N). \tag{1}$$

Roughly, (1) implies that the simpler $\pi_*(R)$ is *homologically*, the simpler the theory of R-modules will be. For example, if $\pi_*(R)$ is of finite *homological dimension*, then the spectral sequence (1) has only finitely many rows at the E_2-page. As a result, one should expect the theory of R-modules to be relatively algebraic.

Example 1. Suppose R is an even periodic $\mathbb{E}_\infty$-ring with $\pi_*(R)$ regular noetherian. Then by [1] the *Picard group* of R arises entirely from algebra. If $\pi_0 R$ has no nontrivial idempotents, then it is $\mathbb{Z}/2 \times \mathrm{Pic}(\pi_0 R)$, the former factor coming from suspensions.

Example 2. Suppose R is an even periodic $\mathbb{E}_\infty$-ring with $\pi_*(R)$ regular noetherian. Then the *thick subcategories* of the category of perfect R-modules are in bijection with the specialization-closed subsets of $\mathrm{Spec}\pi_0 R$.

However, without these hypotheses other things can happen:

Example 3 ([9, Prop. 2.4.9])**.** There exists an $\mathbb{E}_\infty$-ring R with $\pi_i(R) = 0$ for $i \neq 0, -1$, $\pi_0(R) \simeq \mathbb{Q}[\epsilon]/\epsilon^2$, and $\pi_{-1}(R)$ a free $\pi_0(R)$-module of rank one, such that the Picard group of R is given by $\mathbb{Z} \times \mathbb{Q}$.

A number of important $\mathbb{E}_\infty$-ring spectra in modern algebraic topology have the property that, while the homotopy groups are quite complicated, they are related to $\mathbb{E}_\infty$-ring spectra with much simpler homotopy groups. When combined with descent theory, this yields techniques for understanding invariants of them.

1. Descent via thick subcategories

The following is our key definition.

Definition 1. Let $R \to R'$ be a morphism of $\mathbb{E}_\infty$-ring spectra. Then R' is *descendable* as an R-module if the thick tensor ideal[1] that R' generates in $\mathrm{Mod}(R)$ is all of $\mathrm{Mod}(R)$.

Date: March 5, 2015.

[1]Recall that a stable subcategory of $\mathrm{Mod}(R)$ is called *thick* if it is additionally closed under retracts, and a *tensor ideal* if it is closed under tensor products with objects of $\mathrm{Mod}(R)$.

Closely related ideas have been discussed by Balmer in [2].

Theorem 1 (M. [5]). *Suppose $R \to R'$ is descendable. Then one has a canonical equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}(R) \simeq \mathrm{Tot}\left(\mathrm{Mod}(R') \rightrightarrows \mathrm{Mod}(R' \otimes_R R') \substack{\to\\ \to\\ \to} \dots\right).$$

Theorem 1 holds for a faithfully flat morphism $R \to R'$ of $\mathbb{E}_\infty$-rings by the derived version of faithfully flat descent, due to Lurie.[2] However, the main interest of Theorem 1 is that it applies in cases where $\pi_*(R')$ is much simpler than $\pi_*(R)$. For example, it applies to the complexification map $KO \to KU$ (this was previously known to Gepner-Lawson and Meier). For TMF, one obtains the following result.

Theorem 2 (M.-Meier [7]). *There is an equivalence of symmetric monoidal ∞-categories between modules over TMF (resp. Tmf) and quasi-coherent sheaves on the derived moduli stack of elliptic curves (resp. its compactification).*

Using Theorem 2, we were able to stratify thick subcategories of the category of perfect TMF-modules in [6].

Theorem 3 (M. [6]). *The thick subcategories of the category of perfect TMF-modules are in natural bijection with the specialization-closed subsets of the underlying space of the moduli stack of elliptic curves.*

2. Galois theory

An application of this approach to descent theory is a reformulation of the Galois theory of Rognes [11].

Definition 2. Let $\phi : R \to R'$ be a morphism of $\mathbb{E}_\infty$-rings. ϕ is said to be a *finite cover*[3] if there exists a descendable morphism $R \to \widetilde{R}$ such that $\widetilde{R} \to R' \otimes_R \widetilde{R}$ exhibits $R' \otimes_R \widetilde{R}$ as a finite product of copies of $\widetilde{R}$ localized at an idempotent.

Theorem 4 (M. [5]). *Suppose $\pi_0 R$ has no nontrivial idempotents. Then the ∞-category of finite covers of R is a Galois category in the sense of Grothendieck, and therefore has associated to it a Galois group.*

Let $\pi_1(\mathrm{Mod}(R))$ be the Galois group of the $\mathbb{E}_\infty$-ring R. Then a continuous homomorphism $\pi_1(\mathrm{Mod}(R)) \to G$, for G a finite group, corresponds to giving a faithful G-Galois extension of R in the sense of Rognes.

Using descent, one can compute Galois groups in several examples. For instance, the Galois group of integral TMF is trivial; this is equivalent to the folklore algebraic theorem that the étale fundamental group of the moduli stack of elliptic curves over $\mathbb{Z}$ is trivial. If G is a finite p-group and k a separably closed field of characteristic p, the Galois group of the Tate construction k^{tG} is the quotient of G by the normal subgroup generated by the order p elements.

3. Applications to Picard groups

Let R be an $\mathbb{E}_\infty$-ring. Another basic invariant of R is the *Picard group* of invertible R-modules. The Picard group is essentially an algebraic invariant of R if R is connective, or if $\pi_*(R)$ is periodic but sufficiently nice (e.g., even periodic with $\pi_0(R)$ regular noetherian). However, the Picard group can be somewhat mysterious in general for general non-connective $\mathbb{E}_\infty$-rings.

Theorem 5 (M.-Stojanoska [9]; Hopkins (unpublished, for TMF)). *The Picard group of the periodic integral TMF is cyclic, $\mathbb{Z}/576$ generated by ΣTMF. The Picard group of the non-connective, non-periodic Tmf is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/24$.*

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[2] In general, we do not know if a faithfully flat morphism $R \to R'$ satisfies descendability (even in the case of a morphism of discrete commutative rings) without, e.g., countable presentation hypotheses.

[3] The analog of a finite étale morphism.

This is proved by using the equivalence of Theorem 2. In particular, one may compute the Picard group of the derived moduli stack itself. Over an affine scheme étale over the moduli stack, the Picard group is algebraic. To globalize, one obtains a *descent spectral sequence*; the bulk of [9] consists of analyzing this descent spectral sequence in the cases of TMF and Tmf.

4. Vista: Descent in equivariant stable homotopy theory

A current project related to this circle of ideas focuses on applications of descent and thick subcategory ideas in equivariant stable homotopy theory. This project is joint with Niko Naumann and Justin Noel, and was inspired by the work of Carlson [3] and the recent preprint of Balmer [2] interpreting Quillen's classical work on the spectrum of an equivariant cohomology ring [10].

Fix a finite group G and let Sp_G denote the ∞-category of G-spectra.

Definition 3. Let M be a G-spectrum and let $\mathcal{F}$ be a family of subgroups of G. We say that M is **$\mathcal{F}$-nilpotent** if it belongs to the thick tensor ideal generated by $A_{\mathcal{F}} \overset{\text{def}}{=} \prod_{H \in \mathcal{F}} (G/H)_+$.

Theorem 6 (M.-Naumann-Noel [8])**.** *Let R be a ring G-spectrum and let X be any G-space. Suppose R is $\mathcal{F}$-nilpotent for a family $\mathcal{F}$ of subgroups. Consider the Green functor $\{R_H^*(X)\}_{H \subset G}$. Then we have:*

(1) The Green functor $\{R_H^*(X)\}_{H \subset G}$ satisfies a form of Artin induction, i.e.

$$R_G^*(X)[1/|G|] \simeq \varprojlim_{\mathcal{O}_{\mathcal{F}}(G)^{op}} R_H^*(X)[1/|G|],$$

where $\mathcal{O}_{\mathcal{F}}(G)$ is the subcategory of the orbit category of G spanned by the G-sets $\{G/H\}_{H \in \mathcal{F}}$.

(2) The Green functor $\{R_H^*(X)\}_{H \subset G}$ satisfies a version of Quillen's $\mathcal{F}$-isomorphism theorem. That is, the map $R_G^*(X) \to \varprojlim_{\mathcal{O}_{\mathcal{F}}(G)^{op}} R_H^*(X)$ has nilpotent kernel, and anything in the codomain raised to a sufficiently divisible power is in the image.

The Borel equivariant mod p cohomology spectrum is $\mathcal{F}$-nilpotent for $\mathcal{F}$ the family of *abelian subgroups.* This is (essentially) Carlson's and Balmer's intepretation of Quillen's theorem. One of our main results in [8] is that the equivariant spectrum representing a Borel-equivariant *complex-oriented* cohomology theory is always nilpotent for the family of *abelian subgroups.* Other important examples include equivariant real and complex K-theory (for the family of cyclic subgroups) and equivariant elliptic cohomology in the sense of Lurie [4] (for the family of rank ≤ 2 abelian subgroups). Finally, we are studying these nilpotence questions for equivariant *algebraic* K-theory, which we believe will have applications to the descent problem; this is closely related to forthcoming work of Clausen.

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