Descent for ring spectra and applications

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Introduction

- Let R be a commutative ring. One of the basic ways we can study R is to study structures over R: for instance, R-modules.
- One of the basic tools that one has in studying R-modules is Grothendieck's theory of **faithfully flat descent.** Given a faithfully flat R-algebra R', this states that the data of an R-module M is equivalent to that of its base-change $R' \otimes_R M$ with appropriate "descent data."
- Given an R'-module N, a **descent datum** on N is an equivalence of $R' \otimes_R R'$-modules $R' \otimes_R N \simeq N \otimes_R R'$ satisfying a cocycle condition.

Grothendieck's theorem

Theorem (Grothendieck)

If $R \to R'$ is a faithfully flat morphism of commutative rings, then there is a canonical equivalence between the category of R-modules and the category of R'-modules with descent data.

Example

To give a vector space V over the real numbers $\mathbb{R}$ is equivalent to giving a vector space W over $\mathbb{C}$ (here W is the extension of scalars $W = V \otimes_{\mathbb{R}} \mathbb{C}$) together with a $\mathbb{C}$-antilinear map $\iota : W \to W$ with $\iota^2 = \mathrm{id}_W$.

The purpose of this project is to describe an analog of faithfully flat descent with ring spectra, but for morphisms that do not superficially appear to be faithfully flat.

Ring spectra and module spectra

- In stable homotopy theory, the analog of a ring is called a **ring spectrum.**
- Just as a spectrum gives rise to a cohomology theory on spaces, a ring spectrum gives rise to a cohomology theory on the category of topological spaces with **cup products**.
- Let R be an associative (A_∞) ring spectrum. In this case, there is a theory of R-module spectra. [2, 4].
- If R is given a commutative (E_∞) structure, then one can in addition tensor R-modules and one gets a symmetric monoidal product on R-modules [2, 4].
- Many important objects in modern algebraic topology have this type of additional structure.

Working with R-module spectra

Let R be an A_∞-ring spectrum. How does one work with R-module spectra? There are several basic tools one has.

- Given a ring spectrum R, one has its *homotopy groups* $\pi_*(R)$, which form a graded, associative ring.
- Given an R-module M, the first invariant that one sees are its homotopy groups $\pi_*(M)$, which are a graded module over $\pi_*(R)$.
- Given R-modules M, N, to understand the space of R-module maps $M \to N$, there is a spectral sequence

$$\mathrm{Ext}^{s,t}_{\pi_*(R)}(\pi_*(M), \pi_*(N)) \implies \pi_{t-s}\mathrm{Hom}_R(M, N).$$

- Upshot: the simpler that $\pi_*(R)$ is homologically, the easier it will be to work with R-module spectra.

Examples

1. **Complex K-theory** KU is an example of an E_∞-ring spectrum whose homotopy groups are homologically simple: $\pi_*(KU) \simeq \mathbb{Z}[\beta^{\pm 1}]$ where β has degree two.
2. As a result, it is possible to classify perfect KU-modules in terms of algebra; the Ext spectral sequence degenerates.
3. However, a closely related E_∞-ring spectrum, **real** K**-theory** KO, has much more complicated homotopy groups

$$\pi_*(KO) \simeq \mathbb{Z}[t^{\pm 1}, \eta]/(2\eta, \eta^3) \quad \text{where} |t| = 8, |\eta| = 1.$$

4. Homologically, the homotopy groups of $\pi_*(KO)$ have infinite dimension, and so the Ext spectral sequence is much more complicated here.

Descent theory

Let $R \to R'$ be a morphism of E_∞-ring spectra. There is a notion of faithful flatness that makes reference only to the homotopy groups of R and R'.

Definition

R' is **faithfully flat** over R if:

1. $\pi_0(R) \to \pi_0(R')$ is a faithfully flat morphism of (ordinary) commutative rings.
2. The map $\pi_*(R) \otimes_{\pi_0(R)} \pi_0(R') \to \pi_*(R')$ is an isomorphism.

The faithfully flat descent theorem

Definition

Let $R \to R'$ be a morphism of E_∞-ring spectra. The **cobar construction** is the cosimplicial E_∞-ring

$$R' \rightrightarrows R' \otimes_R R' \substack{\to \\ \to \\ \to} \dots$$

and the **∞-category of descent data** is the totalization of the cosimplicial ∞-category

$$\mathrm{Tot}\left(\mathrm{Mod}(R') \rightrightarrows \mathrm{Mod}(R' \otimes_R R') \substack{\to \\ \to \\ \to} \dots\right).$$

Theorem (Lurie)

Let $R \to R'$ be a faithfully flat morphism of E_∞-ring spectra. Then the ∞-category of R-modules is equivalent to the ∞-category of R'-modules with descent data.

KO and KU

1. This does not help us if we want to study KO-modules. Any KO-algebra faithfully flat over KO has to have homotopy groups of infinite homological dimension, too.
2. A **key observation** (due to Rognes, Gepner-Lawson, Meier, and others) is that there exists a morphism of E_∞-ring spectra $KO \to KU$ which behaves like a C_2-torsor in ordinary algebraic geometry (like $\mathbb{R} \to \mathbb{C}$).

Theorem (Gepner-Lawson, Meier)

There is a C_2-action on the symmetric monoidal, ∞-category $\mathrm{Mod}(KU)$ of KU-module spectra whose homotopy fixed points are given by $\mathrm{Mod}(KO)$.

3. As a result, it is possible (Hopkins) to use techniques of **Galois descent** to work with KO-modules, e.g., to compute the **Picard group**.

Thick subcategories and tensor-ideals

Definition

Let $\mathcal{C}$ be a triangulated category (or a stable ∞-category). A full subcategory $\mathcal{C}' \subset \mathcal{C}$ is **thick** [3] if:

1. $\mathcal{C}'$ is closed under cofibers and desuspensions (i.e., is a triangulated or stable subcategory).
2. $\mathcal{C}'$ is closed under retracts. Equivalently, if $X \oplus Y \in \mathcal{C}'$, then $X, Y \in \mathcal{C}'$.

Suppose $\mathcal{C}$ is given a symmetric mononoidal structure. Then $\mathcal{C}'$ is called a **thick tensor-ideal** if whenever $X \in \mathcal{C}'$ and $Y \in \mathcal{C}$, then $X \otimes Y \in \mathcal{C}'$.

Descendability

Let $R \to R'$ be a morphism of E_∞-ring spectra. Then R' defines an object in $\mathrm{Mod}(R)$.

Definition (Balmer [1]; M. [5])

The map $R \to R'$ is **descendable** if the thick tensor-ideal that R' generates in R-modules is the entire ∞-category of R-modules.

This is equivalent to the condition that the cobar construction from $R \to R'$ not only converges to R, but should define a **constant pro-object.**

Examples of descendable morphisms

Example

1. Faithfully flat extensions **under countability hypotheses.**
2. Quotienting by a nilpotent ideal in ordinary commutative algebra (e.g., $\mathbb{C}[\epsilon]/\epsilon^2 \to \mathbb{C}$).
3. (Carlson) Let G be a p-group and k a field of characteristic p. Then the map

$$k^{BG} \to \prod_{A \subset G \text{ elementary abelian}} k^{BA}$$

   is descendable.
4. (Hopkins-Ravenel) The map $L_n S^0 \to E_n$, where E_n is Morava E-theory and L_n denotes E_n-localization is descendable.
5. $KO \to KU$ (in fact, any **faithful Galois extension** in the sense of Rognes).

It is important that these examples (except the first) are **not faithfully flat.**

Main result

Theorem (M.)

The conclusion of faithfully flat descent holds for a descendable morphism of E_∞-ring spectra. That is, there is an equivalence

$$\mathrm{Mod}(R) \simeq \mathrm{Tot}\left(\mathrm{Mod}(R') \rightrightarrows \mathrm{Mod}(R' \otimes_R R') \dots\right).$$

As a result, one obtains numerous examples of non-faithfully flat descent in stable homotopy theory. Often, when $\pi_*(R)$ is homologically complicated and $\pi_*(R')$ is simpler, the ∞-category of descent data is actually much easier to work with.

Applications

These ideas have several applications to the understanding of certain invariants of structured ring spectra.

1. A new point of view on Rognes's Galois extensions [8] and a formulation in terms of axiomatic Galois theory (as well as several computations of Galois groups) in [5].
2. Calculations of Picard groups of topological modular forms using descent spectral sequences (joint with V. Stojanoska [7]).
3. Classifications of thick subcategories of modules over ring spectra such as *TMF* [6].
4. Generalizations of Quillen's F-isomorphism theorem to equivariant cohomology theories (joint work in progress with N. Naumann and J. Noel).

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