Mixed characteristic commutative algebra

Thm CM: Fix a prime p

1) Local: R excellent noeth domain, $p \in \mathrm{Rad}(R)$

$\Rightarrow$ $R^+ :=$ absolute int closure of R

R^+/p is CM over R/p

$\left((R^+)^\wedge \text{ is CM over } R \text{ under mild conditions} \right)$

2) Global: $V =$ p-adic DVR, X/V proper flat scheme of rel dim d,

$L \in \mathrm{Pic}(X)$ semiample + big

Then $\exists$ a finite cover $\pi : Y \longrightarrow X$ s.t

$$H^{<d}(X_{p=0}, L^{-1}) \longrightarrow H^{<d}(Y_{p=0}, L^{-1})$$

is 0 $\left(\text{where } X_{p=0} = X \otimes_{\mathbb{Z}} \mathbb{Z}/p \right)$

Rmk: 1) Thm CM (1) gives an explicit construction of "weakly functorial CM algebras" (André, Gabber)

2) Thm CM (2) has applications to MMP in mixed characteristic, ex: with MPSTWW, we show that flip exist in dim 3.

3) Related ideas + (rational) p-adic Hodge theory $\Rightarrow$

Given X/V as above, the following two subsheaves of $\omega_{X[1/p]/V[1/p]}$ are equal:

- $\mathcal{I}(\omega_{X[\frac{1}{p}]}) = \pi_* \omega_Z$ for any resolution $\pi: Z \to X[\frac{1}{p}]$

- $$\tau(\omega_X)\left[\tfrac{1}{p}\right] = \left(\bigcap_{\substack{Y \xrightarrow{\pi} X \\ \text{finite covers} \\ \text{dom by } X^+}} \mathrm{Im}\left(\pi_* \omega_Y \xrightarrow{tr} \omega_X\right)\right)\left[\tfrac{1}{p}\right]$$

Pf of Thm CM in the almost category

Thm ACM: $C/\mathbb{Q}_p$ complete + alg closed.

$R/\mathcal{O}_C$ fin. presented flat domain

$\Rightarrow R^+/p$ is almost CM over R/p

(i.e. $H^i_x((R^+/p)_x) \overset{a}{=} 0 \quad \forall\, i < \dim((R/p)_x)$

$\forall\, x \in \operatorname{Spec}(R/p)$)

Sketch of proof:

Say $X = \operatorname{Spec}(R[\frac{1}{p}]) \,/\, C$

$X^+ = \operatorname{Spec}(R^+[\frac{1}{p}]) \xrightarrow{\pi} X$

As R^+ is perfectoid after p-completion,

Thm RH $\Rightarrow$ $RH(\pi_* \mathbb{F}_p) \overset{a}{=} R^+/p$

$\therefore$ it suffices to show:

a) $\pi_* \mathbb{F}_p$ is ind-perverse (up to shift):

$$\pi_* \mathbb{F}_p = \operatorname{colim} F_i[-n], \quad \text{where}$$

$F_i \in \mathrm{Perv}(X; \mathbb{F}_p)$ and $n = \dim(X)$

b) F perverse $\Rightarrow$ $RH(F)[-n]$ is almost CM $/R/p$

<u>Pf of (a)</u>: The Key Lemma from last time implies that $\underline{\mathbb{F}_p}_{X^+}$ is $*$-extended from the gen pt of X^+

$$\therefore \pi_* \underline{\mathbb{F}_p}_{X^+} = \operatorname*{colim}_{\substack{Y \xrightarrow{\pi_Y} X \\ \text{finite covers}}} \pi_{Y,*}(\eta_{Y,*} \underline{\mathbb{F}_p})$$

$\therefore$ suffices to observe that

$$\pi_{Y,*}(\eta_{Y,*} \mathbb{F}_p) = \pi_{Y,*}\Big(\operatorname*{colim}_{U \overset{j}{\subset} Y} j_* \mathbb{F}_p\Big)$$

and $\pi_{Y*} j_* \underline{\mathbb{F}_p}[n]$ is perverse for $j: U \subset Y$ being a smooth affine open (Artin).

<u>Pf of (b)</u> : $\mathcal{F}$ perverse $\Rightarrow$

$$RH(\mathcal{F}) = \mathbb{D}\, \underbrace{RH(\mathbb{D}(\mathcal{F}))}_{\text{Corrective}} \in \mathbb{D}_{Groth}(D^{\leq 0})$$

$\mathbb{D}(\mathcal{F})$: perverse

$\therefore$ suffices to show: $\mathbb{D}_{Groth}(D^{\leq 0}) \in \{\text{almost CM complexes}\}$

This follows from the proof of:

<u>Lemma</u>: X noeth scheme admitting dualizing complex

$$\Rightarrow \mathbb{D}_{Groth}(D^{\leq 0})[-\dim X] \in \{\text{CM complex}\}$$

$$= \{K \mid H^i_x(K_x) \in D^{\geq \dim(\mathcal{O}_{X,x})} \ \forall x \in X\}$$

$\blacksquare$

<u>Rmk</u>: 1) Thm ACM previously known in dim ≤ 3 (Heitmann)

2) In part (a), can show $\pi_* \mathbb{F}_p[n] = \bigcup_{\substack{\pi_Y : Y \to X \\ \text{finite}}} \pi_{Y,*} \underline{IC}_Y$

Aside: How does RH relate to $\mathcal{D}$-modules?

$K/\mathbb{Q}_p$ disc valued field, $C = \hat{\bar{K}}$

X_0/K proper variety, $X = X_0 \otimes_K C$

Expect:

$$\begin{tikzcd}
\mathrm{Perv}_{WHT}(X_0; \mathbb{Q}_p) \arrow[r, "\widetilde{RH}"] \arrow[dd, "RH"] & \left\{ \begin{array}{c} \text{good filtered + regular} \\ \text{holonomic } \mathcal{D}_{X_0}\text{-mods} \end{array} \right\} \arrow[d, "gr^*"] \\
 & D^b_{coh}(T^*X_0) \arrow[d, "0_X^*"] \\
D^b_{coh}(X) & D^b_{coh}(X_0) \arrow[l]
\end{tikzcd}$$

Thm CM in the real world

To prove Thm CM, we follow the outline of

Thm HL (Huneke-Lyubeznik):

R = excellent noeth. local domain $/\mathbb{F}_p$

$\exists$ a finite cover $R \to S$ s.t

$$H^i_m(R) \longrightarrow H^i_m(S) \quad \text{is } 0$$

$\forall\ i < \dim(R)$

Pf Strategy:

1) Induction : Using induction on dim, $\exists$ a finite cover $R \to S$ s.t

$H^i_m(R) \longrightarrow H^i_m(S)$ has finite length image for $i < \dim(R)$

2) Killing cohomology: For any finite length Frob-stable submodule $M \in H^i_m(R)$, $\exists$ $R \to S$ fin cover killing M.

In mixed char, we follow same argument using Δ instead of Θ

Outline of pf of Thm CM

$V = \overline{\mathbb{Z}_p}$, $X = \mathbb{A}^n_V$, $x \in X_{p=0}$ closed

$(A, (d))$ perfect prism s.t $A/(d) = V$

<u>Goal</u>: $H^i_x\left(\mathcal{O}_X^+/p\right) = 0 \quad \forall\ i < n$ $(*)$

$\overline{K} = \overline{K(x)}$

$$P = \left\{ \begin{tikzcd} & Y \arrow[d, "\pi_Y"] & \leftarrow \text{semistable}/V \\ \operatorname{Spec}(\overline{K}) \arrow[ur] \arrow[r] & X & \end{tikzcd} \quad \leftarrow \text{alteration} \right\}$$

<u>Obs</u>: Thm DS from last time $\Rightarrow$

$$\operatorname*{colim}_{Y \in P} R\pi_{Y,*}\, \mathcal{O}_Y/p = \mathcal{O}_X^+/p$$

$\therefore (*) \iff \operatorname*{colim}_{Y \in P} H^i_x\left(R\pi_{Y*}\, \mathcal{O}_Y/p\right) = 0 \quad i < n$

(Log) Hodge - Tate comp theorem

$$\Longleftrightarrow \quad \operatorname*{colim}_{Y \in P} H^i_x\left(\Delta_Y / (p,d)\right) = 0 \quad \forall\, i < n$$

where $\Delta_Y = \underline{\underline{\log}}$ Δ-coh of Y (Česnavičius-Koshikawa)

$$\overset{\text{Bockstein}}{\Longleftrightarrow} \quad \operatorname*{colim}_{Y \in P} H^i_x\left(\Delta_Y / p\right) = 0 \quad \forall\, i < n+1$$

$$\circlearrowleft \phi$$

Proof of via the Huneke-Lyubeznik method

1) Killing cohomology: Any Frob-stable $M \subset H^i_x(\Delta_Y/p)$ which is f.g $/V$ is killed by some $Y' \to Y$ in P.

2) Induction on dim: For any $Y \in P$, $\exists$ $Y' \to Y$ in P s.t.

$$H^i_x\left(\Delta_Y/(p,d)\right) \longrightarrow H^i_x\left(\Delta_{Y'}/(p,d)\right)$$

has fg image $\forall\, i < n$

3) Bounding d-torsion : $\forall\, Y \in P$, $\exists$ a $Y' \to Y$ in P s.t. the image of

$$H^i_x(\Delta_Y/p) \longrightarrow H^i_x(\Delta_{Y'}/p)$$

is d^c-torsion for some $c = c(n)$, $i < n+1$.

4) Combine (2) + (3) : $\forall\, Y \in P$, $\exists\, Y' \to Y$ s.t. $\mathrm{im}\left(H^i_x(\Delta_Y/p) \longrightarrow H^i_x(\Delta_{Y'}/p)\right)$ is f.g. $/V$ $\forall\, i < n+1$

5) (1) + (4) $\Rightarrow$ (*') . $\blacksquare$