A p-adic Riemann-Hilbert functor for $\mathbb{F}_p$-coefts (11/9)

(joint with Lurie)

<u>Last time</u>: $C/\mathbb{Q}_p$ complete + alg closed field, $k = \mathcal{O}_C/\mathfrak{m}$

R = p-complete $\mathcal{O}_C$-algebra

$$\rightsquigarrow \quad R_{\mathrm{pfd}} \in \mathrm{CAlg}\left(D^{\geq 0}_{\mathrm{comp}}(R)\right) \quad \text{s.t.}$$

a) $R_{\mathrm{pfd}} \otimes_{\mathcal{O}_C} k \cong (R \otimes_{\mathcal{O}_C} k)_{\mathrm{perf}}$

b) If $R/\mathcal{O}_C$ is formally smooth, then

$$R_{\mathrm{pfd}}[\tfrac{1}{p}] \xleftarrow{\sim} \overline{\Delta}_R[\tfrac{1}{p}] \xrightarrow{\sim} \bigoplus_i \Omega^i_{R/\mathcal{O}_C}[-i]\underbrace{\{-i\}}_{\ker(\theta)^{-i}}[\tfrac{1}{p}]$$

isogeny theorem

Hodge-Tate comparison (depends on a choice of a lift of $R[\frac{1}{p}]$ to $A_{\inf}/\xi^2[\frac{1}{p}]$)

<u>Today</u>: Build a Riemann-Hilbert functor using R_{pfd} as a coherent counterpart of $\mathbb{Z}_{p,X}$

Setup: $C/\mathbb{Q}_p$ as above

$X/\mathcal{O}_C$ any scheme

$X_0 = X \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$ scheme $/\mathcal{O}_C/p$

Recall: Recall that the max ideal $\mathfrak{m} \subseteq \mathcal{O}_C$ satisfies

$$\mathfrak{m} \overset{L}{\underset{\mathcal{O}_C}{\otimes}} \mathfrak{m} \xrightarrow{\sim} \mathfrak{m}.$$

$\Rightarrow$ restriction of scalars $D(\mathcal{O}_C/\mathfrak{m}) \longrightarrow D(\mathcal{O}_C)$

is fully faithful

$\therefore$ $D(\mathcal{O}_C)^a := D(\mathcal{O}_C)/D(\mathcal{O}_C/\mathfrak{m})$ $\leftarrow$ almost derived category

Likewise: $D_{qc}(X_0)^a$ = almost quasi-coherent derived category of X_0

$$\left(= D_{qc}(X_0)\Big/\left\{ M \in D_{qc}(X_0) \;\middle|\; M \text{ almost zero over } \mathcal{O}_C \right\}\right)$$

Have a semiorthogonal decomp.

$$D_{qc}(X_k) \longrightarrow D_{qc}(X_0) \longrightarrow D_{qc}(X_0)^a$$

should be $X_0 \overset{L}{\underset{\mathcal{O}_C}{\otimes}} k$

Classically, we have

$$D(X_k, \mathbb{F}_p) \xrightarrow{i_*} D(X, \mathbb{F}_p) \xrightarrow{j^*} D(X_C, \mathbb{F}_p)$$

<u>Thm</u>: $\exists$ a colimit preserving functor

$$RH : D(X, \mathbb{F}_p) \longrightarrow D_{qc}(X_0)^{\phi=1}$$

$\{ M \in D_{qc}(X_0) +$
$M \to \phi_* M \}$

with the foll. features:

a) Normalization: $RH(\underline{\mathbb{F}_p}_X) = \mathcal{O}_{X,\text{perfd}}/p$

b) Proper pushforwards: If $f : Y \to X$ is proper,

$$RH \circ Rf_* \cong Rf_* \circ RH$$

Now assume $X/\mathcal{O}_C$ is fin. presented + flat

Then RH is fully faithful and satisfies:

c) Almost coherent: If $F \in D^b_{cons}(X; \mathbb{F}_p)$, then

$RH(F)$ is <u>almost coherent</u> $/X_0$, <u>i.e</u> $\forall\, \varepsilon \in \mathfrak{m}$,

$\exists$ a coherent $N_\varepsilon \in D^b_{coh}(X_0)$ + $N_\varepsilon \longrightarrow RH(F)$ with

cone killed by ε.

d) Duality: On the subcategory $D^b_{cons}(X_C, \mathbb{F}_p)$
$$\underset{!\text{-ext}}{\cap|}$$
$$D(X, \mathbb{F}_p)$$
,

$$RH \circ \mathbb{D}_{Verdier} \overset{a}{\simeq} \mathbb{D}_{Groth} \circ RH$$

e) Perverse: $RH({}^p D^{\leq 0}(X_C, \mathbb{F}_p)) \overset{a}{\subseteq} D^{\leq 0}_{qc}(X_0)$

<u>Rmk</u>: 1) Special fibre: If $p=0$ on X, get a covariant version of RH for schemes of char p

2) Almost coherence: inspired by Zavyalov's thesis

<u>Ex</u>: $\bigoplus_{n \geq 0} \mathcal{O}_C/p^{1/n}$ is almost fg over $\mathcal{O}_C$

$\prod_{n \geq 0} \mathcal{O}_C/p^{1/n}$ " " " " "

3) RH is lax symm monoidal, and thus gives
$$\widetilde{RH}: D(X, \mathbb{F}_p) \longrightarrow D_{qc}(X_0, \mathcal{O}_{X,\text{perfd}}/p)$$
This is symmetric monoidal and compatible with pullbacks

4) $\exists$ a version with mod p^n coeffs of (3)

Take a limit, invert p, to get:

$$\begin{tikzcd}
D(X_C, \mathcal{O}_p) \arrow[r] \arrow[dr] & D_{qc}\left(\hat{X}[\tfrac{1}{p}], \mathcal{O}_{X,\mathrm{pfd}}[\tfrac{1}{p}]\right) \arrow[d, equal, "X/\mathcal{O}_C \text{ smooth}"] \\
& D_{qc}\left(\hat{X}[\tfrac{1}{p}], \bigoplus_i \Omega^i_{X_C/C}[-i]\{-i\}\right)
\end{tikzcd}$$

<u>Sketch of construction</u>:

- define by hand (using (a) + (b)) for very large $\mathcal{O}_C$-schemes
- rest by arc descent

<u>Application</u>

<u>Thm DS</u>: R noeth ~~ring~~ domain, X/R proper scheme

Then $\exists$ a finite cover $\pi : Y \to X$ s.t.

$R\Gamma(X, \mathcal{O}_X/p) \longrightarrow R\Gamma(Y, \mathcal{O}_Y/p)$ factors over

$$H^0(Y, \mathcal{O}_Y)/p \longrightarrow R\Gamma(Y, \mathcal{O}_Y/p)$$

Rmk: 1) This was previously known:

- R has char p
- $\dim(R) = 1$: in Beilinson's proof of

C_{dR} (geom input of p-adic P.L)

2) Thm implies:

R noeth $\mathbb{Z}_p$-scheme $p \in \mathrm{Rad}(R)$

Assume R is a <u>splinter</u> (i.e. any finite inj map

$R \to S$ has an R-linear splitting)

Then R has "rational singularities"

Ex: $R = \mathbb{Z}_p[x,y]/(x^3+y^3+p^3)$ $\leftarrow$ 2 dim normal local domain

$$\begin{array}{ccc}
X = \mathrm{Bl}_0(\mathrm{Spec}(R)) & \xrightarrow{\pi} & \mathrm{Spec}(R) \\
\uparrow \text{ regular } \cup\!\mid & & \cup\!\mid \\
\mathbb{P}^2_{\mathbb{F}_p} \supset E = V(x^3+y^3+z^3) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_p)
\end{array}$$

Can show $H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(E, \mathcal{O}_E) \cong \mathbb{F}_p \neq 0$

<u>Exercise</u>: Prove Thm for this example

<u>Pf of Thm DS</u>:

May assume X, R integral, $X \to \mathrm{Spec}(R)$ dominant

Let $Y \longrightarrow X$ be an absolute integral closure of X

By limit arguments, need to show

$$(*) \quad R\Gamma(Y, \mathcal{O}_Y/p) \xleftarrow{\sim} H^0(Y, \mathcal{O}_Y)/p$$

Have a diagram

$$f^+_* \mathcal{O}_Y \to \begin{tikzcd} Y \arrow[r] \arrow[d, "f^+"] & X \arrow[d, "f"] \\ \mathrm{Spec}(R^+) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

$\shortparallel$

R^+

abs. integral closure of R

Both $\hat{Y}$ and $\hat{R^+}$ are perfectoid (easy exercise)

By proper pushforward compat of RH, we have

$Rf^+_* \mathcal{O}_Y/p = Rf^+_* \mathcal{O}_{Y,\mathrm{pfd}}/p = RH(Rf^+_* \mathbb{F}_p)$

$\therefore$ $(*)$ is implied by

$(*')$: $Rf^+_* \underline{\mathbb{F}_p} \cong \underline{\mathbb{F}_p}$

This follows from

<u>Lemma</u> : Y integral normal scheme with alg closed function field. A = ab group. Then

$$R\Gamma(Y_{et}, \underline{A}) \xleftarrow{\sim} A$$

<u>Pf</u> : • $Y_{et} = Y_{zar}$

• Groth's thm $R\Gamma(\underset{\text{space}}{\text{irred}}, \underline{A}) = \underline{A}$ ∎