A p-adic Riemann-Hilbert functor and vanishing theorems

I) The characteristic p story

Recall: a noeth local ring (R, m) is CM

if it satisfies:

a) Every system of parameters is a regular sequence

b) $H^i_m(R) = 0 \quad \forall i < \dim(R)$

c) Say $X = \mathrm{Spec}(R)$. For each pt $i_x : \{x\} \hookrightarrow \mathrm{Spec}(R)$

we have $i_x^! \mathcal{O}_X \in D^{\geq \dim(\mathcal{O}_{X,x})}$

d) (If R admits a dualizing complex)

$\omega_R^\bullet$ is concentrated in a single deg.

Examples: regular rings, rational singularities $/\mathbb{C}$

F-rational sings $/\mathbb{F}_p$, ...

Thm HH (Hochster-Huneke)

1) (R, m) excellent noeth local $\mathbb{F}_p$ - domain

R^+ = absolute int closure

= integral closure of R in $\overline{\mathrm{Frac}(R)}$

Then R^+ is CM, i.e. $H^i_m(R^+) = 0$

$\forall i < \dim(R)$

2) Say k is field of char p

X/k proj variety, L ample l.b on X

Then $\exists$ a finite surj map $\pi : Y \longrightarrow X$

s.t π^* kills:

- $H^{>0}(X, L^a)$ $a \geq 0$
- $H^{< \dim(X)}(X, L^b)$ $b < 0$

Ex: $X = \mathrm{Cone}(A \subset \mathbb{P}^n) \subset \mathbb{P}^{n+1}$

Then $H^{<\dim}(X, \mathcal{O}(-1)) \neq 0$ over any field (☆)

Rmk: 1) Thm HH (1) is false in char 0

as soon as $\dim(R) \geq 3$ (due to trace obstructions)

2) Thm HH (i) $\iff$ R excellent regular $\mathbb{F}_p$-algebra, then $R \to R^+$ is (faithfully) flat

<u>Example</u>: Say X/k proj variety, $k = \bar{k}$ char p

Consider $H^i(X, \mathcal{O}_X)$ for some $i > 0$

$\circlearrowleft_{\mathrm{Frob}}$

p-linear algebra $\Rightarrow$

Frob nilp elts

$$H^i(X, \mathcal{O}_X) \cong H^i(X, \mathcal{O}_X)_{\mathrm{nilp}} \oplus \left(H^i(X, \mathcal{O}_X)^{\mathrm{Frob}=1} \otimes_{\mathbb{F}_p} k \right)$$

Now $H^i(X, \mathcal{O}_X)_{\mathrm{nilp}}$ is killed by Frob^n $\forall n \gg 0$

Also, $H^i(X, \mathcal{O}_X)^{\mathrm{Frob}=1} \simeq H^i(X, \mathbb{F}_p)$ Artin-Schreier

<u>Exercise</u>: Y any qcqs scheme, $i > 0$

each $\alpha \in H^i(Y, \mathbb{F}_p)$ is killed by a finite cover of Y. ∎

A sample application (Hochster–Roberts)

Say S is smooth $\mathbb{C}$-algebra, G red. group

$G \curvearrowright S \rightsquigarrow R = S^G \subset S$

<u>Thm</u>: R is CM

By the Reynolds operator, $R \hookrightarrow S$ is R-linear summand

By spreading out, <u>Thm</u> reduces to

$(*)$ $R \to S$ is an R-linearly split inclusion of noeth. excellent $\mathbb{F}_p$-algebras with S regular

$\Rightarrow R$ is CM

<u>Pf of $(*)$</u>: Consider the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R^+ \arrow[d] \\ S \arrow[r] & S^+ \end{tikzcd}$$

split by assumption

split as an S-module map by Thm HH for S

$\Rightarrow$ $R \to R^+$ is R-linearly split

Thm HH for R $\Rightarrow$ R^+ is CM, so R is CM $\blacksquare$

Other results one can prove similarly:

1) Direct summand conj
2) Faltings connectedness thm
3) Kollár's conj or improving local Lefschetz from SGA2

Goal of this series: mixed char analog of Thm HH

Plan: 1) Use (perfect) prismatic cohomology to build a Riemann-Hilbert functor for $\mathbb{F}_p$-sheaves on varieties / p-adic fields

2) Use (1) to "almost" accomplish Goal

3) Use (imperfect, log) prismatic coh to upgrade (2) to an honest result

II) Review of (perfect) prismatic cohomology
(joint with Morrow, Scholze)

Def: A perfect prism is a pair (A, I) where

- $A = W(S)$, S perfect $\mathbb{F}_p$-alg. $\circlearrowleft \phi$

- $I \subset A$ ideal "primitive of deg 1", i.e.

$I = (d)$ s.t. $d = \sum_{i \geq 0} [d_i] \cdot p^i$, we have

$d_1 \in S^*$, S is d_0-adically complete

Ex: 1) $A = W(S)$ for S perfect of char p

$I = (p)$ "crystalline perfect prisms"

2) Perfect q-dR prism:

$$A = \mathbb{Z}_p [q^{1/p^\infty}]^\wedge_{(p, q-1)} \cong W\left(\mathbb{F}_p [q^{1/p^\infty}]^\wedge_{(q-1)}\right)$$

$I = ([p]_q)$, where $[n]_q = \frac{q^n - 1}{q-1}$

Note: $A/I = \mathbb{Z}_p^{cycl}$ = ring of integers of $\mathbb{Q}_p(\mu_{p^\infty})^\wedge$

$q^{1/p^n} \mapsto \varepsilon_n$ = primitive p^n-th root of 1

3) $C/\mathbb{Q}_p$ complete + alg closed field

Fontaine: $A_{inf}(\mathcal{O}_C) = W(\varprojlim_\phi \mathcal{O}_C/p) \xrightarrow{\theta} \mathcal{O}_C$

$(A_{inf}(\mathcal{O}_C), \ker(\theta))$ is a perfect prism

<u>Rmk</u>: $\{\text{perfect prisms}\} \cong \{\text{perfectoid rings}\}$

$$(A, I) \longmapsto A/I$$

<u>Thm</u>: Fix (A, I) perfect prism

R = p-complete A/I-algebra

$\exists$ a naturally defined comm. algebra object

$$\Delta_R \in D_{comp}(A)$$ always (p, I)-adic completions

+ $\phi_{R/A} : \phi_A^* \Delta_R \longrightarrow \Delta_R$ satisfying.

a) Hodge-Tate: $\exists$ an increasing mult filt on

$$\overline{\Delta}_R = \Delta_R \overset{L}{\underset{A}{\otimes}} A/I \quad \text{s.t}$$

$$gr_*(\overline{\Delta}_R) \cong \Lambda^* L_{R/(A/I)}[-*]$$

b) de Rham: $(\phi_A^* \Delta_R)/I \cong L\Omega_{R/(A/I)}$

derived dR complex (p-completed)

c) Étale : $\left(\Delta_R/p \left[\frac{1}{I}\right] \right)^{\phi=1} \simeq R\Gamma\left(\mathrm{Spec}\left(R[\tfrac{1}{p}]\right), \mathbb{F}_p\right)$

d) Isogeny : R smooth $/ (A/I)$, then

$$\phi_{R/A} : \phi_A^* \Delta_R \longrightarrow \Delta_R$$

is an I-isogeny

<u>Construction</u>: For a p-complete A/I-alg R, set

$$\Delta_{R,\mathrm{perf}} = \left(\varinjlim_{\phi} \Delta_R \xrightarrow{\phi} \Delta_R \xrightarrow{\phi} \Delta_R \cdots \right)^\wedge \in D_{\mathrm{comp}}(A)$$

$$R_{\mathrm{perfd}} := \left(\Delta_{R,\mathrm{perf}}\right)/I \in D_{\mathrm{comp}}(R)$$

perfectoidization of R

<u>Rmk</u>. 1) If $I = (p)$, then $R_{\mathrm{perfd}} \simeq R_{\mathrm{perf}} := \varinjlim_{\mathrm{Frob}} R$

2) We always have $R_{\mathrm{perfd}} \in D^{\geq 0}$

<u>If</u> R_{perfd} is in deg 0, then

R_{perfd} is a perfectoid and $R \longrightarrow R_{perfd}$

is the universal map to a perfectoid

Example: Say (A, I) is perfect q-dR prism

so $A/I = \mathbb{Z}_p^{cycl}$

Say $R = A/I\,[x^{\pm 1}]^\wedge$. One can show:

$\phi_A^* \Delta_R$ = "q-de Rham complex of R/A"

$$= A[x^{\pm 1}]^\wedge \xrightarrow{\nabla_q} A[x^{\pm 1}]^\wedge \cdot d\log_q(x)$$

$$= \widehat{\bigoplus_{i \in \mathbb{Z}}} \left(A \cdot x^i \xrightarrow{[i]_q} A \cdot x^i \, d\log_q(x) \right)$$

Using this, one shows:

$$R_{pfd} = \widehat{\bigoplus_{i \in \mathbb{Z}[\frac{1}{p}]}} \left(A/I \cdot x^i \xrightarrow{q^i - 1} \mathfrak{m}\, A/I \cdot x^i \right)$$

where $\mathfrak{m} \subset A/I = \mathbb{Z}_p^{cycl}$ is the max ideal $\left(\text{note: } \mathfrak{m}/\mathfrak{m}^2 = 0\right)$

In particular, $H^1(R_{pfd}) \neq 0$