GROUP HOMOLOGY

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In the following, $K$ is always an abelian group and $Q$ is any group. In addition, $K$ is a $Q$-module: it comes with a homomorphism $Q \to \text{Aut}(K)$. In other words, we can think of elements of $Q$ as being invertible homomorphisms $K \to K$ which compose according to the group law of $Q$.

A short exact sequence, written

$$0 \to K \xrightarrow{i} E \xrightarrow{p} Q \to 0$$

is an injective homomorphism $i: K \to E$ and a surjective homomorphism $p: E \to Q$ such that the kernel of $p$ equals the image of $i$.

(1) Show that in any short exact sequence $0 \to K \to E \to Q \to 0$, if all of the groups are finite then $\#K + \#Q = \#E$.

(2) Suppose that you are given a short exact sequence $0 \to K \to E \xrightarrow{p} Q \to 0$ and a homomorphism $t: Q \to E$ such that $pt = 1_Q$. Prove that $E \cong Q \rtimes K$.

(3) Suppose that we have a short exact sequence

$$0 \to K \to E \to Q \to 0.$$

Suppose that the action of $Q$ on $K$ given by this short exact sequence is nontrivial, so that the homomorphism $Q \to \text{Aut}(K)$ is not a constant homomorphism. Show that $E$ is not abelian.

(4) Let $Z^2(Q, K)$ be the set of functions $c: Q \times Q \to K$ such that for all $q_1, q_2, q_3 \in Q$,

$$q_1 \cdot c(q_2, q_3) - c(q_1 q_2, q_3) + c(q_1, q_2 q_3) - c(q_2, q_3) = 0.$$

Show that $Z^2(Q, K)$ is an abelian group with operation given by $(c_1 + c_2)(q_1, q_2) = c_1(q_1, q_2) + c_2(q_1, q_2)$.

(5) Let $B^2(Q, K)$ be the set of functions $g: Q \times Q \to K$ which are of the form $g(q_1, q_2) = f(q_1) + q_1 \cdot f(q_2) - f(q_1 q_2)$ for some function $f: Q \to K$. Show that $B^2(Q, K)$ is a group which is a subgroup of $Z^2(Q, K)$.

(6) Classify all exact sequences of the form

$$0 \to \mathbb{Z} \to E \to \mathbb{Z}/3 \to 0.$$

(Hint: first, explain why the action of $\mathbb{Z}/3$ on $\mathbb{Z}$ must be trivial.)

(7) Classify all exact sequences of the form

$$0 \to \mathbb{Z} \to E \to \mathbb{Z}/2 \to 0.$$