1. Lecture 1, June 28th

1.1. History. How do we define length?

\[ \mathbb{N} \quad \mathbb{Q} \quad \mathbb{R} \]

1 \quad \frac{1}{2} \quad \frac{\sqrt{2}}{2} \quad 1

How do we define areas of shapes? If it is a polygon, we can cut it into many pieces and reassemble them into a rectangle with one side equal to 1. See Figure 1. What about a circle?? Well we can approximate it with polygons. See Figure 2.

**Figure 1.** How to cut a polygon into a rectangle with one side equal to 1.

**Figure 2.** How to approximate a circle with polygons.
Here's an attempted algorithm to show that if two polygons have the same area, then they can be rearranged into one another.

1. Cut the first polygon into triangles. (See Figure 3.)
2. Rearrange every triangle into a rectangle. (See Figure 4.) Note that if your triangle is not acute, the base needs to be the edge corresponding to the non-acute angle.
3. Rearrange every rectangle into a rectangle with one side equal to 1. See the following claim.
4. Combine these rectangles.
5. Do the same for the second polygon. Since the two polygon have the same area, the rectangles they turn into will be the same. Combining the rearrangement of the first one with the reversed rearrangement of the second one (i.e., taking the common cuts), we can rearrange the first polygon into the second polygon.

Claim 1.1. Every rectangle can be rearranges into a rectangle with one side equal to 1

*Proof.* (See Figure 5) Arrange the two rectangles as in the figure, where the side $AG$ has side 1. Without loss of generality, assume the midpoint of $AB$ lies in the segment $AD$. (If not, there is a way to double the length of $AD$ and half the length of $AC$, by cutting the rectangle into half vertically and reassemble them horizontally.)

$$\Box ADEC = ADIHG + \triangle CGH + \triangle ECI$$
$$\cong ADIHG + \triangle IDB + \triangle FHB$$
$$= \Box ABFG$$

The readers should check that $\triangle CGH \cong \triangle ECI$ and $\triangle IDB \cong \triangle FHB$. 
Now we have shown that if two polygons have the same area, then they can be rearranged into one another. Another question arises.

**Question 1.2.** What if we only allow some class of isometries?

**Definition 1.3.** An *isometry* on $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that preserves the metric. Two polygons are *congruent* if there is an isometry taking one to the other.

**Exercise 1.4.** (See Henry’s handout on Hyperbolic Geometry) Show that all isometries on $\mathbb{R}^2$ are compositions of translations, reflections, and rotations.

**Question 1.5.** Can we arrange every polygon into any other polygons only using translations?

**Answer 1.6.** No, rotations by $180^\circ$ is necessary.

**Claim 1.7.** If two polygons have the same area, they can be rearranged into one another using translations and rotations by $180^\circ$.

**Proof.** Note that we only need translations for steps (1) and (3). For step (2) we need translations and rotations by $180^\circ$. For step (4) it seems like we need other rotations to line up the rectangles, but it turns out that we don’t!!! See Figure 6.

**Definition 1.8.** (Hadwiger Invariants) (See Figure 7.) For every direction in $\mathbb{R}^2$, take edges that are perpendicular to the direction chosen, take the length of these edges, with a plus sign if the direction is pointing outward, a minus sign if the direction if pointing inward. The sum of these numbers is the Hadwiger invariant with respect to this direction. Hadwiger invariants are the collection of these values with respect to all possible directions.
Claim 1.9. If two polygons can be rearranged into one another using only translations, then they have the same Hadwiger invariants. In particular, the rectangle and the triangle in Figure 8 cannot be rearranged into one another only using translations. (Clearly they can be rearranged into one another using one rotation by $180^\circ$.)

Proof. (See Figure 9.) When we cut a polygon into two polygons, there are three cases. In the yellow polygon, it is clear that the Hadwiger invariant along the direction is not changed at all. In the green polygon, an edge of length $l$ is split into two edges of length $l', l''$, respectively, but the Hadwiger invariant remains unchanged since $l = l' + l''$. In the blue polygon, we have two new edges perpendicular to the direction, with the same length but opposite signs. Hence they add up to zero. □

Cool Fact 1.10. If two polygons have the same area and all Hadwiger invariants are the same, then they can be rearranged into one another using translations.

Definition 1.11. Polygons $P$ and $Q$ are scissors congruent, denoted $P \sim Q$, if we can write

1. $P = \bigcup_{i=1}^{n} P_i$
2. $Q = \bigcup_{i=1}^{n} Q_i$

such that $P_i$'s and $Q_i$'s are polygons, $P_i \cong Q_i, i = 1, ..., n$, and $P_i \cap P_j$ and $Q_i \cap Q_j$ have measure zero if $i \neq j$, i.e., just overlap on boundary. We say that $P$ and $Q$ are equidecomposable.

Definition 1.12. We say $P, Q$ are stably scissors congruent if there exist $R, S$ with $R \sim S$ such that $P \sqcup R \sim Q \sqcup S$. 
Figure 7. How to define Hadwiger invariants.

Figure 8. Proof of Claim 1.9.
Remark 1.13. Stable scissors congruence does NOT imply scissors congruence in general. See Figure 10. In the hyperbolic plane, the triangles $P \parallel R$ and $Q \parallel R$ are congruent. However, $P$ and $Q$ are not scissors congruent, because $Q$ contains a point on the boundary, while $P$ does not. See Henry’s notes on hyperbolic geometry to know more about the hyperbolic plane.

**Theorem 1.14.** (Zylev) In the Euclidean plane, stable scissors congruence implies scissors congruence.

**Proof.** In construction. □
Definition 2.1. Two polygons are scissors congruent if we can write $P = \bigcup_{1 \leq i \leq n} P_i$, $Q = \bigcup_{1 \leq i \leq n} Q_i$ such that

1. $P_i \cong Q_i$ as polygons (related by a rigid motion), $i = 1, \ldots, n$
2. $P_i \cap P_j$ and $Q_i \cap Q_j$ have measure zero for $i \neq j$

Question: What are 3-dimensional scissors congruence invariants?

(1) Volume
Gauss said if you are allowed infinitely many pieces, then volume is the ONLY invariant (BORING)

Theorem 2.2. Two prisms are scissors congruent if and only if they have the same volume.

Sketch Proof. See Figure 11. Look straight down and pretend that they are 2-dimensional; rearrange polygons into $1 \times L$ rectangles. We now know that the prisms look like Figure 12. Turn these so that 1 is the height. See Figure 13. Looking down, we have the rectangles with the same area so we can rearrange them into one another. This can also work if the prism is at an angle, but you have to be careful. Instead, we can turn a slanted prism into a rectangular one — slice it perpendicular to the line of extension (think of the parallelogram case in 2 dimensions).

2.1. Primer to Group Theory.

Definition 2.3. A group is a set $G$ together with a binary operation $\star$ and an element $e$ (the identity) that satisfies

1. $a \star e = e \star a = a$ for any $a \in G$
2. $(a \star b) \star c = a \star (b \star c)$
3. For any $a \in G$ there exists an element $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.

Example 2.4. (1) $\mathbb{Z}$, $+, 0$
(2) $\mathbb{Z}/n\mathbb{Z}$ — modular arithmetic
(3) $\mathbb{Q}^\times$ with multiplication and 1
(4) $\mathbb{R}^\times$ with multiplication and 1
(5) $\Sigma_n$ — permutations of $n$ letters, i.e. bijections of the set $\{1, \ldots, n\}$ (this group is noncommutative)
(6) Cyclic group - $e, a, a \star a, a \star a \star a, \ldots, a^{-1}, a^{-1} \star a^{-1}, \ldots$. The groups 1 and 2 above are cyclic; we can show that these groups are the only examples.

Definition 2.5. A group $G$ is abelian if $a \star b = b \star a$ for all $a, b \in G$ (commutative).

Sometimes it’s easier to study maps between objects than the objects themselves!

Definition 2.6. A homomorphism $\phi : (G, \star, e) \rightarrow (H, \ast, e')$ is a function such that for all $a, b \in G$, $\phi(a \star b) = \phi(a) \ast \phi(b)$

The point is that a homomorphism preserves group structure! Some consequences:

Exercise 2.7. Check that $\phi(e) = e'$ and $\phi(a)^{-1} = \phi(a^{-1})$.

Exercise 2.8. The map $\phi(a) = a^{-1}$ is a homomorphism if and only if $G$ is abelian.
2.1.1. Tensor Products.

**Definition 2.9.** Let $G$ and $H$ be abelian groups. $G \otimes H := \{ \sum_{i=1}^{\text{finite}} a_i (g_i \otimes h_i) : g_i \in G, h_i \in H, a_i \in \mathbb{Z} \}$ modulo the equivalence relations:

1. $a_1(g \otimes h) + a_2(g \otimes h) = (a_1 + a_2)(g \otimes h)$
2. $(g_1 * g_2) \otimes h = g_1 \otimes h + g_2 \otimes h$
3. $g \otimes (h_1 * h_2) = g \otimes h_1 + g \otimes h_2$

**Example 2.10.** What is $\mathbb{Z} \otimes \mathbb{Z}$? Well, using the relations above: $m \otimes n = (1 + \ldots + 1) \otimes n = 1 \otimes n + \ldots + 1 \otimes n = m(1 \otimes n) = m(1 \otimes 1 + \ldots + 1 \otimes 1) = mn(1 \otimes 1)$. So therefore $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$

**Exercise 2.11.** What is $\mathbb{Z}^2 \otimes \mathbb{Z}^2$? $\mathbb{Z}_p \otimes \mathbb{Z}_q$? Let $p$ and $q$ be distinct primes, $\mathbb{Z}_p \otimes \mathbb{Z}_q$?
Example 2.12. Consider the group \( \mathbb{R}/\pi \mathbb{Z}, +, 0 \). As a set, this is real numbers up to division by \( \pi \) (so that \( r \sim b \) if \( r - b \) is divisible by \( \pi \)).

Example 2.13. What we are really interested in is \( \mathbb{R} \otimes \mathbb{R}/\pi \mathbb{Z} \). This group is HARD, just like geometry.

References given:
(1) Artin’s Algebra
(2) Dummit and Foote’s Abstract Algebra

back to our regularly scheduled problem:

2.2. Dehn Invariants. Dehn defined the following map in 1901:
\[ \Delta : \mathcal{P} = \{ \text{Polyhedra} \} \to \mathbb{R} \otimes \mathbb{R}/\pi \mathbb{Z}, P \mapsto \sum_{\text{edges of } P} \text{length}(e) \otimes (\text{dihedral angle at } e) \]

We need to show that the map is constant on scissors congruent classes. To do this, we can pretend that \( \mathcal{P} \) is a group. So we can define \( \mathcal{P} = \{ \text{formal sums of polyhedra} \} / \sim \) where the equivalence relations are \( nP + mP \sim (n + m)P \), \( P = P' + P'' \) if \( P = P' \cup P'' \) and \( P' \cap P'' \) have measure zero, and \( P = Q \) if \( P \) and \( Q \) are congruent. The group operation is + and 0 is the empty polyhedra.

If \( P, Q \) are scissors congruent, then \([P] \in \mathcal{P}\) is equal to \([P_1] + ... + [P_n] = [Q_1] + ... + [Q_n] = [Q]\). Note that in this group scissors congruent polyhedra are identified.

Theorem 2.14 (Zylev). \( P \) is scissors congruent to \( Q \) if and only if \([P] = [Q]\) in \( \mathcal{P} \).

Volume is a scissors congruent invariant is exactly saying that we have a homomorphism: \( \text{Vol} : \mathcal{P} \to \mathbb{R} \). We want to say that \( \Delta \) above is also a homomorphism. Let us check this.

Proof.
(1) \( \Delta(P + P) = 2\Delta(P) \).
This is trivial.

(2) Congruence.
The rigid motion that takes \( P \) to \( Q \) gives a bijection between the edges of \( P \) to the edges of \( Q \) such that the angle at \( e \) in \( P \) is the same as the angle at \( e \) in \( Q \).

(3) \( \Delta(P) = \Delta(P') + \Delta(P'') \) if \( P = P' \cup P'' \) and \( P' \cap P'' \) have measure zero.
Suffices to check this for slicing \( P \) by a plane.

What happens if we sum over the edges of \( P' \) and \( P'' \)? Purple edges cancel out nicely but green edges are sliced across by lanes and split into 2 edges with the same angle. On the left hand side this will be \( L \otimes \theta \) and on the right hand side this will be \( L' \otimes \theta + L'' \otimes \theta \), each in \( P' \) and \( P'' \) respectively. But we know that \( L = L' + L'' \) so just use properties of the tensor product!

Along the orange edge, the length does not change but the angle split in two. You get \( L \otimes \theta = L \otimes \theta' + L \otimes \theta'' \) and same argument works!

Lastly, there will be new edges (pink). The angles around new edges add up to \( \pi \) — since the homomorphism lands in \( \mathbb{R}/\pi \mathbb{Z} \), these guys will not contribute.

□

Theorem 2.15. If \( P \) is a prism, then the Dehn invariant of \( P \) is 0.
Proof. Assume that the prism is orthogonal. (See Figure). Because ends are parallel, angles at corresponding edges add up to π so $\Delta(P) = |P_1| \otimes (\theta_1) + ... + |P_n| \otimes (\theta_n) + |P'_1| \otimes (\theta_1) + ... + |P'_n| \otimes (\theta_n) + |e_1| \otimes (\psi_1) + ... + |e_n| \otimes (\psi_n) = |e_1| \otimes (\Sigma \text{angles}) = 0$. □

Let $a, b \in (0, 1)$. Let $\alpha, \beta \in (0, \pi/2)$ where $\sin^2 \alpha = a, \sin^2 \beta = b$. Define $T(a, b)$ to be a tetrahedron with vertices $(0, 0, 0), (\cot \alpha, 0, 0), (\cot \alpha, \cot \alpha \cot \beta, 0), (\cot \alpha, \cot \alpha \cot \beta, \cot \beta)$. The upshot of this is that all of these faces are right triangles.

Fact: any polyhedron can be cut into such tetrahedra. We call one of these $T(a, b)$ an orthoscheme.

Define $\alpha \star \beta := \text{angle such that } \sin^2(\alpha \star \beta) = ab$ where angle at “long edge” is $\alpha \star \beta$. We can check that $\Delta(T(a, b)) = \cot \alpha \otimes \alpha + \cot \beta \otimes \beta - \cot(\alpha \star \beta) \otimes (\alpha \star \beta)$ which is nonzero.

3. Lecture 3, July 5th

Recall that we defined the group $\mathcal{P}$ which are formal sums of polyhedra. Each element can be written as $a_1P_1 + ... + a_nP_n$ and they are subject to some relations. We recall that Zylew’s theorem just means that the classes here are equivalent if and only if they are scissors congruent.

We have an addition in this set: $P + Q = P \cup Q$ if $P \cap Q$ has measure zero (in order for volume to be invariant). We can only add some polyhedra but not all.

(PICTURE1 - cube and tetrahedron)

Notice at this point that we are only allowed to add, but not subtract. What lets us do subtraction is if we allow the coefficients in our formal sums to have negatives. This process is called group completion. Sometimes when you do that bad things happen.

(PICTURE2 - adding finite segment to infinite segment)

Zylew’s theorem says that nothing goes wrong when you take group completion.

3.1. Back to the Dehn Invariant. We have constructed two homomorphisms $Vol : \mathcal{P} \to \mathbb{R}$ and $\Delta : \mathcal{P} \to \mathbb{R} \otimes \mathbb{R}/\pi \mathbb{Z}$. Recall that if $P$ and $Q$ are prisms and
Figure 15. How to cut an arbitrary prism into an orthogonal prism.

Figure 16. Proof of Theorem 2.15.

volP = volQ then P is scissors congruent to Q, and if P is a prism then Δ(P) = 0.

Let's now construct a polyhedra with a nonzero Dehn invariant.

(PICTURE - Dehn Invariant)

Σ_{edges} \text{len}(e) \otimes \theta(e) = Σ_{edges} a \otimes \theta(e) = 6(a \otimes \theta(e)).

Now we need the fact that:

Lemma 3.1. If x ∈ \mathbb{R} \otimes \mathbb{R}/\pi \mathbb{Z} is nonzero, then so is nx for all \(0 \neq 0 \in \mathbb{Z}.

Now we want to show that \(\alpha \otimes \cos^{-1} 1/3 \neq 0\) but it suffices to show that \(1 \otimes \cos^{-1} 1/3 \neq 0\). We know that \(\cos k\pi = \pm 1\). Furthermore, by definition, \(\cos \alpha = 1/3, \sin \alpha = 2\sqrt{2}/3\).
Figure 17. How to cut any tetrahedron into 24 orthoschemes (only two of them are drawn in the picture), where $I$ is the incenter of the tetrahedron.

Figure 18. $T\left(\frac{1}{2}, \frac{1}{4}\right)$. 
Figure 19. This picture shows that $\cos (\theta (e)) = \frac{1}{3}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\cos (n\alpha)$</th>
<th>$\sin (n\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{2\sqrt{2}}{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{-7}{3}$</td>
<td>$\frac{4\sqrt{2}}{3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{-23}{27}$</td>
<td>$\frac{-10\sqrt{2}}{27}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{17}{81}$</td>
<td>$\frac{-56\sqrt{2}}{81}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

When we compute this, note that the denominators are always powers of 3, and the numerator of cos’s are integers and the sin’s are an integer times $\sqrt{2}$. Use induction to prove this. It turns out that the numerator of cos’s are $1 (mod 3)$ if $k$ is odd and $2 (mod 3)$ if $k$ is even and the reverse happens (times $\sqrt{2}$) of course for the sin case. So the conclusion is that $\cos k\alpha$ is not $\pm 1$ (try doing this yourself). This means the Dehn invariant is thus nonzero!

3.2. Action of $\mathbb{R}$. Given any real number $\alpha$, there exists a map $\tilde{\alpha} : \mathcal{P} \to \mathcal{P}$ that scales by $\alpha$. Note that $\tilde{\alpha} \circ \tilde{\mu} = \tilde{\alpha \circ \mu}$. We can also do this by negatives and just stick in a negative sign.

Now, $vol(\tilde{\lambda}(P)) = \lambda^3 vol(P)$. However, here’s a nifty statement: $\Delta(\tilde{\lambda}(P)) = \lambda \Delta(P)$ (think about this).
3.3. Algebra Timeout - Quotients.

Definition 3.2. Let $G$ be an abelian group, $H$ a subgroup of $G$ (a subset of $G$ such that it is closed under group operation and inverses), define $G/H := \{g + H : g \in G\}$ with operation: $(g_1 + H) + (g_2 + H) = (g_1 + g_2) + H$ (check that this is well-defined and is indeed a group operation!)

Example 3.3. $G = \mathbb{Z}, H = n\mathbb{Z}$, then $\mathbb{Z}/n\mathbb{Z}$ is the integers modulo $n$.

Example 3.4. If $\phi : G \to H$ is a homomorphism, then $\ker \phi = \{g \in G : \phi(g) = 0\}$. It is a fact that $G/\ker \phi \cong \text{im} \phi$.

3.4. Back to Action of $\mathbb{R}$. We have one homomorphism that behaves well with respect to $\lambda$ and another that does not. To rectify this, we kill prisms.

Let $\mathcal{E}$ be the subgroup of $\mathcal{P}$ generated by prisms. In other words, formal sums of $a_1R_1 + \ldots + a_nR_n$ where the $R_i$’s are prisms. We want to look at $\mathcal{P}/\mathcal{E}$. So in this group, two polyhedra are the same if you tack in a prism to each one.

Theorem 3.5. $\mathcal{P}/\mathcal{E}$ is a real vector space.

Proof. We need to check that we can add elements — but this is true because we already have a group. Now we need to be able to multiply by real numbers — this is again true by the action above.

Next, $(\lambda_1 \lambda_2)x = \lambda_1(\lambda_2)x$ but this is again true. The one you should be excited about is $(\lambda_1 + \lambda_2)x = \lambda_1x + \lambda_2x$. Proving this is the point of modding out prisms. The point is that the difference of the sums above is exactly 2 prisms. Look at the artwork on the paper on the website!

The point is that vector spaces are so much more restrictive than just groups. Another interesting fact:

Proposition 3.6. $\Delta$ factors through $\mathcal{P}/\mathcal{E}$. This is to say that there exists a unique map $\delta$ such that the following diagram commutes:
Theorem 3.7 (Sydler 1965). If $P$ and $Q$ have the same Dehn invariant and the volume then they are scissors congruent.

To prove this, we want to show that $\delta$ is injective.

Proof. Suppose that $\text{vol}(P) = \text{vol}(Q)$, $\Delta(P) = \Delta(Q)$. Because $\delta$ is injective, $[P] = [Q]$ means that there exists prisms $R, S$ such that $P \cup R$ is scissors congruent to $Q \cup S$ which means that $\text{vol}(P \cup R) = \text{vol}(Q \cup S)$ which means that $\text{vol}(R) = \text{vol}(S)$ which means that $R$ is scissors congruent to $S$ and thus $P$ is scissors congruent to $Q$. By Zylev’s theorem, we have that $P$ is scissors congruent to $Q$.

3.5. Algebraic Rephrasing.

Definition 3.8. A homomorphism $\phi : \mathbb{R} \to P/\mathcal{E}$ if $\phi(\pi) = 0$ and for all $a, b \in (0, 1)$, the class of $[T(a, b)] \in P/\mathcal{E}$ is equal to $\sum_{\text{edges}T(a, b)} \text{len}(e) \phi(\theta(e))$.

By homomorphism we mean that $\phi(a + b) = \phi(a) + \phi(b), \phi(na) = n\phi(a)$ for integers $n$. This is weird, really weird. What we are really doing is to construct a homomorphism back from $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$ to $P$ and pray for an inverse.

Proposition 3.9. If a good function exists, then Sydler’s theorem holds.

Proof. Define $\Phi : \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z} \to P/\mathcal{E}$ such that $\Phi \circ \delta = \text{id}$. This is easily defined by $\Phi(x \otimes y) = x\phi(y)$. This is a well-defined function (check). The intuition is to restrict to the classes $[T(a, b)]$’s which generates everything (to be shown later).

Suppose that we have a function, $h : (0, 1) \to P/\mathcal{E}$ such that $[T(a, b)] = h(a) + h(b) - h(a, b)$ and, if $a + b = 1$, then $ah(a) + bh(b) = 0$. Call this a homological function. Then define $\phi(\alpha) = \tan \alpha \cdot h(\sin^2 \alpha), \phi(n\pi/2) = 0$. Then $\phi$ is good.