Math 258 Midterm #1

Name: ________________________________

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Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
1. In this question, \( F = \mathbb{R} \) and \( V = \mathbb{R}^3 \).

   (a) (5 points) What is the dimension of the kernel of
   \[
   \begin{pmatrix}
   1 & 4 & 6 \\
   2 & 3 & -1 \\
   3 & 10 & 5 \\
   \end{pmatrix}
   \]

   **Solution:** We proceed with row reduction.

   \[
   \begin{pmatrix}
   1 & 4 & 6 \\
   2 & 3 & -1 \\
   3 & 10 & 5 \\
   \end{pmatrix}
   \overset{R_2 - 2R_1, R_3 - 3R_1}{\longrightarrow}
   \begin{pmatrix}
   1 & 4 & 6 \\
   0 & -5 & -13 \\
   0 & -2 & -13 \\
   \end{pmatrix}
   \overset{R_3 - R_2}{\longrightarrow}
   \begin{pmatrix}
   1 & 4 & 6 \\
   0 & -2 & -13 \\
   0 & 3 & 0 \\
   \end{pmatrix}
   \]

   Thus the rank of the transformation is 3, and the dimension of the kernel is 0.

   (b) (5 points) What are the eigenvalues and eigenvectors of
   \[
   \begin{pmatrix}
   \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\
   0 & \pi & 0 \\
   \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\
   \end{pmatrix}
   \]

   You may not use determinants in your solution.

   **Solution:** Note that \((0, 1, 0)\) is an eigenvector with eigenvalue \(\pi\). The subspace spanned by \((1, 0, 0)\) and \((0, 0, 1)\) is invariant under the transformation, so all other eigenvalues and eigenvectors must be in this subspace. But the transformation restricted to this subspace is a rotation by \(\pi/6\), so it has no eigenvalues. Thus \(\pi\) is the only eigenvalue and it has eigenvector \((0, 1, 0)\).

2. Let \( F \) be a field, and suppose that we are given a polynomial
   \[
   p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x].
   \]

   We define a ring \( F[x]/p(x) \) to be the set
   \[
   \{ b_{n-1}x^{n-1} + \cdots + b_1x + b_0 \mid b_0, \ldots, b_{n-1} \in F \}.
   \]

   Addition and multiplication work the same way as for polynomials, with the extra rule that
   \[
   x^n = -a_{n-1}x^{n-1} - \cdots - a_1x - a_0.
   \]
(a) (5 points) Show that there exists an injective ring homomorphism \( F \to F[x]/p(x). \)

**Solution:** For all \( \alpha \in F, \) take it to the vector \( 0x^{n-1} + \cdots + 0x + \alpha. \) Since constant polynomials multiply like constants, this is an injective field homomorphism.

(b) (10 points) Suppose that we can write \( p(x) = a(x)b(x) \) in \( F[x], \) with \( \deg a(x), \deg b(x) \geq 1. \) Show that there exist \( \alpha, \beta \in F[x] \) such that \( \alpha \beta = 0. \)

**Solution:** Suppose that \( \deg a(x) = m. \) Then we can write \( a(x) = c_m x^m + \cdots + c_1 x + c_0 \) and \( b(x) = d_{n-m} x^{n-m} + \cdots + d_1 x + d_0 \) with \( 1 \leq m, m - n \leq n - 1. \) Let

\[
\alpha = 0x^{n-1} + \cdots + 0x^{m+1} + c_m x^m + \cdots + c_1 x + c_0 \\
\beta = 0x^{n-1} + \cdots + 0x^{n-m+1} + d_{n-m} x^{n-m} + \cdots + d_1 x + d_0.
\]

Then if we multiply \( \alpha \) and \( \beta \) by the rules of multiplying polynomials we get \( p(x); \) however, inside \( F[x]/p(x) \) we must rewrite the \( x^n \) term. Then we get

\[
\alpha \beta = (-a_{n-1} x^{n-1} - \cdots - a_1 x - a_0) + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,
\]

as desired.

(c) (10 points) Show that there exists \( \alpha \in F[x]/p(x) \) such that

\[
\alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0.
\]

**Solution:** Let

\[
\alpha = 0x^{n-1} + \cdots + 0x^2 + x + 0.
\]

Then \( \alpha^n = x^n = -a_{n-1} x^{n-1} - \cdots - a_1 x - a_0. \) Therefore

\[
\alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 \\
= (-a_{n-1} x^{n-1} - \cdots - a_1 x - a_0) + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\
= 0.
\]

(d) (10 points) Suppose that \( F = \mathbb{R} \) and \( p(x) = x^2 + 1. \) In this case the ring \( \mathbb{R}[x]/p(x) \) is actually one of the standard fields that we discussed. Which one?

**Solution:** In this case \( \mathbb{R}[x]/p(x) \) is isomorphic to \( \mathbb{C}. \) We take the polynomial \( a_1 x + a_0 \) to \( a_1 i + a_0. \) This is clearly a vector space homomorphism, so we just
need to check that multiplication works properly. We check:

\[(a_1i + a_0)(b_1i + b_0) = a_1b_1i^2 + (a_1b_0 + a_0b_1)i + a_0b_0\]
\[= a_1b_1(-1) + (a_1b_0 + a_0b_1)i + a_0b_0\]
\[= (a_1b_0 + a_0b_1)i + (a_0b_0 - a_1b_1),\]

which is exactly the way multiplication in \(\mathbb{C}\) is defined.

3. (20 points) Fix a vector space \(W\) of dim \(W = n > 0\) over \(F\). Let \(V\) be the set of functions \(W \rightarrow W\). Let \(V_e\) be the subspace of even functions (functions such that \(f(x) = f(-x)\)) and \(V_o\) the subspace of odd functions (functions such that \(f(-x) = -f(x)\)). For which fields is it the case that \(V = V_e \oplus V_o\)? Prove that your answer is complete.

**Solution:** We need char \(F \neq 2\). In this case, we can write

\[f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)).\]

Thus if char \(F \neq 2\) then \(V = V_e + V_o\). To see that this sum is a direct sum, we need to check that \(V_e \cap V_o = \{0\}\). Indeed, note that if \(f(x) = f(-x)\) and \(f(x) = -f(-x)\) then \(2f(-x) = 0\); since char \(F \neq 2\) we can divide by 2 and see that \(f(-x) = 0\), as desired.

On the other hand, if char \(F = 2\) then \(V_e = V_o\) since \(-1 = 1\). However, it’s also the case that for all \(x, x = -x\) and \(f(x) = -f(x)\), so \(f(x) = f(-x)\) and \(f(x) = -f(-x)\). Thus \(V_e = V\) as well. Thus the sum is not a direct sum, as any vector \(v \in V\) can be written in at least two ways: \(0 + v\) and \(v + 0\) in \(V_e + V_o\).

4. Suppose that \(V = U \oplus W\). Let \(P\) be the linear transformation that takes a vector \(v \in V\), written as \(v = u + w\) with \(u \in U\) and \(w \in W\), to \(w\).

(a) (10 points) Find all eigenvalues and eigenvectors of \(P\).

**Solution:** Let \(v = u + w\), and suppose that \(v\) is an eigenvalue of \(P\) with eigenvalue \(\lambda\). Then \(P(u + w) = w = \lambda u + \lambda w\), so we get

\[(1 - \lambda)w = \lambda u.\]

Since \(U \cap W = \{0\}\), both sides of this have to be 0. Since at least one of \(u\) and \(w\) is nonzero, we must have either \(\lambda = 0\) or \(\lambda = 1\). If \(\lambda = 0\) then \(w = 0\), so these are just the vectors in \(U\). If \(\lambda = 1\) then \(u = 0\) and we see that these are just the vectors in \(W\).
(b) (5 points) Suppose that $V$ is finite dimensional with $\dim U = m$ and $\dim W = n$. Pick a nice basis of $V$ and write down the matrix of $P$ with respect to that basis.

**Solution:** Let $\{u_1, \ldots, u_m\}$ is a basis of $U$ and $\{w_1, \ldots, w_n\}$ is a basis of $W$. Our basis of $V$ will be $\{u_1, \ldots, u_m, w_1, \ldots, w_n\}$. Then the matrix of $P$ with respect to this basis is

$$
\begin{pmatrix}
0 & & & & \\
& \ddots & & & \\
& & 0 & & \\
& & & 1 & \\
& & & & \ddots \\
& & & & & 1
\end{pmatrix}
$$

where there are $m$ 1’s and $n$ 0’s.

(c) (10 points) Now suppose that $m > 1$. Show that $P$ has an “$m$-th root”: a linear transformation $T : V \to V$ such that $T^m = P$ but $T^i \neq P$ for $i = 1, \ldots, m - 1$.

**Solution:** We define the transformation $T$ on the basis from the above part. We define $T$ by setting

$$
T(u_1) = 0 \quad T(u_i) = u_{i-1} \quad i = 2, \ldots, m
$$

and $T(w) = w$ for $w \in W$. Then $T^m(u_j) = 0$ and $T^m(w) = 1$, so $T^m = P$. On the other hand, $T(u_j) \neq 0$ for $j > 1$ and $P(u_j) = 0$ so $T \neq P$. 