MATH 258 HOMEWORK #6

DUE THURSDAY, FEBRUARY 12

(1) Show that there is a natural (i.e., no choice of basis required) isomorphism $V \otimes W \cong W \otimes V$.

(2) (a) Show that for any vector space $V$, $V \otimes F \cong V$.
(b) Suppose that $V, W, Z$ are vector spaces. Show that there exists a natural (i.e., no choice of basis required) bijection

$$\mathcal{L}(V \otimes W, Z) \longleftrightarrow \mathcal{L}(V, \mathcal{L}(W, Z)),$$

(Hint: If you have trouble understanding what is going on, try showing that for sets $A, B, C$ the set of functions $A \times B \to C$ is in bijection with the set of functions $A \to \text{Func}(B,C)$.)

(c) Using the results of problem (1) and part (b), show that there are bijections

$$\mathcal{L}(V, V^*) \longleftrightarrow \mathcal{L}(V \otimes V^*, F) \longleftrightarrow \mathcal{L}(V^*, V^*).$$

Under these bijections, where does the identity transformation $V^* \to V^*$ go inside $\mathcal{L}(V, V^*)$?

(3) Check that $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is 0 if and only if the vectors $v_1, \ldots, v_k$ are linearly dependent. Use this to show that $\dim (\bigwedge^k V) = \binom{\dim V}{k}$.

(4) Prove that if $T : V \to W$ and $T' : V' \to W'$ are linear transformations then so is $T \otimes T' : V \otimes V' \to W \otimes W'$. If we are also given linear transformations $S : W \to Z$ and $S' : W' \to Z'$ show that

$$(S \otimes S') \circ (T \otimes T') = (S \circ T) \otimes (S' \circ T').$$

In addition, show that $T \wedge T$ is a linear transformation $V \wedge V \to W \wedge W$, and that

$$(S \wedge S) \circ (T \wedge T) = (S \circ T) \wedge (S \circ T).$$

(5) Use the result of the previous problem to show that determinants are multiplicative: for linear transformations $S, T : V \to V$

$$\det(S \circ T) = (\det S)(\det T).$$

(6) Show that our definition of determinant satisfies the definition of a determinant function in Hoffman & Kunze, p. 144. (You’ll need both of the definitions on that page; pay attention also to the paragraph between them.) In the next section they show that determinant functions are unique; thus our definition of the determinant agrees with theirs.

(7) Let $V = \mathbb{R}^3$, and let $\text{vol} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ be the function that takes three vectors $v_1, v_2, v_3$ to the volume of the parallelepiped with vertices $0, v_1, v_2, v_3, v_1 + v_2, v_1 + v_3, v_2 + v_3, v_1 + v_2, v_1 + v_2 + v_3$.

(a) Suppose that $v_1 = v + v'$. When is

$$\text{vol}(v_1, v_2, v_3) = \text{vol}(v, v_2, v_3) + \text{vol}(v', v_2, v_3)?$$

(b) Let $\text{sgn}(v_1, v_2, v_3)$ be 1 if $v_1, v_2, v_3$ satisfy the right-hand rule, and $-1$ otherwise. Show that

$$\text{sgn}(v_1, v_2, v_3) \text{vol}(v_1, v_2, v_3) = \text{sgn}(v, v_2, v_3) \text{vol}(v, v_2, v_3) + \text{sgn}(v', v_2, v_3) \text{vol}(v', v_2, v_3).$$

(c) Explain why $\det T$ is often called the “oriented volume” of $T$. (Hint: look at the definition of a determinant function.)