4a. \[ \lim_{{h \to 0}} \frac{f(1+h) - f(1)}{h} = \lim_{{h \to 0}} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{{h \to 0}} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \]

\[ f'(1) = \frac{1}{2\sqrt{2}} \]

8. It is clear by Proposition 4.7 that \( f(x) \) is differentiable at all points \( x \neq 0 \), so we only need to show that \( f(x) \) is differentiable at \( x = 0 \). First, we check continuity at 0.

\[ \lim_{{x \to 0}} f(x) = 0 \quad \text{and} \quad \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} x^r = 0 \quad \text{so} \]

The left and right limits are equal, and thus \( f(x) \) is continuous at 0.

Now, \[ \frac{f(x) - f(0)}{x - 0} = \begin{cases} 0 & \text{if } x_0 < 0 \\ x_0^{n-1} & \text{if } x_0 > 0 \end{cases} \]

Thus \( \lim_{{x_0 \to 0}} \frac{f(x) - f(0)}{x_0 - 0} \) exists and equal to 0, and so

\[ f \] is differentiable at all \( x \in \mathbb{R} \).
9) \( f(x) \leq x^2 \implies f(0) \leq 0 \implies f(0) = 0 \)

\[
\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{f(x)}{x} \right| \leq |x|
\]

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0, \text{ and hence } f \text{ is differentiable at } x = 0
\]

\[f'(0) = 0\]

10) \( g(1) = 3 \)

\[
\lim_{x \to 1} g(x) = a + b \cdot \lim_{x \to 1} x = a + b
\]

If \( g(x) \) is differentiable at \( x = 1 \), it is continuous, and \( \lim_{x \to 1} g(x) = 3 \)

On the other hand, \( \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} \)

(definition of differentiability)

\( \implies 6 = \frac{a + b}{1} \implies \begin{cases} a = -3 \\ b = 6 \end{cases} \)

14) \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h} = \lim_{h \to 0} f(x_0) - f(x_0) + \lim_{h \to 0} f(x_0) - f(x_0 - h) \)

\( = \lim_{h \to 0} \frac{f(x_0) - f(x_0) + \lim_{h \to 0} f(x_0) - f(x_0 - h)}{h} \)

\( = \lim_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{h \to 0} \frac{f(x_0) - f(x)}{x_0 - x} \)

\( \overset{\text{(i)}}{=} \lim_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \)

\( \overset{\text{(ii)}}{=} \lim_{h \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \)

\( \therefore h \to 0 \iff x \to x_0 \text{ and } x' \to x_0 \)

\( = 2f'(x_0) \)
5) Let \( f(x) = x^5 + 5x + 1 \).

Then \( f(-1) = -5 < 0 \) and \( f(0) = 1 > 0 \).

So by the intermediate value theorem, \( f(x) \) has at least one root between \(-1\) and 0.

\( f'(x) = 5x^4 + 5 > 0 \), so \( f \) is strictly increasing.

Thus \( f \) has exactly one solution between 0 and 1, as desired.

20) \( f(0) = 0 \)

\[
\frac{f(x) - f(0)}{x - 0} = \begin{cases} 
\frac{x - x^2}{x} = 1 - x & \text{if } x \in (0, 1) \\
\frac{x + x^2}{x} = 1 + x & \text{if } x \in (1, \infty)
\end{cases}
\]

Thus \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 1 \).

Given any neighborhood \( I \) of the point 0, assume that I contains the open interval \((-\delta, \delta)\).

Pick an irrational \( 0 < r < \delta \). Then \( f(r) = r + r^2 \).

Since rational numbers are dense, we can find a rational \( s < \delta \).

\( s + \) is \( -r + \delta \). Then \( f(s) = f(r) \)

\[
= -\left(2r^2 + \delta(2r-1) + \delta^2\right).
\]

If \( 2r - 1 \geq 0 \), \( f(s) - f(r) < 0 \) and we would be done.

Otherwise, pick \( \varepsilon = \min(1 - 2r, \delta - r) \), and then \( f(s) - f(r) < 0 \).

Thus, there is no neighborhood \( I \) of the point 0 on which \( f \) is monotonically increasing.

Note: Many think \( f(x) = \begin{cases} 1 - 2x & x \in (0, 1) \\
1 + 2x & x \in (1, \infty) \end{cases} \). This is very not correct.

\( f(x) \) is not a polynomial, so the rules of differentiation don't hold.
15) Let \( g(x) = x^3 \). Then
\[
\lim_{x \to 0} \frac{f(x^3) - f(0)}{x} = \lim_{x \to 0} \frac{f(g(x)) - f(g(0))}{x} = (f \circ g)'(0) \quad \text{by the Chain Rule}
\]
\[
= f'(g(0)) g'(0) = f'(0) \cdot 0 = 0.
\]

Solution to HW 2

4.2.5) \( f^{-1}(f(x)) = x \)

By the Inverse Rule
\[
f^{-1}'(f(x)) f'(x) = 1
\]
But \( f'(x) < 0 \), a contradiction.

Thus \( f^{-1} : f(I) \to \mathbb{R} \) is not differentiable at \( f(x) \).

9) \( f \) odd \(\Rightarrow\) \( f(-x) = -f(x) \) \(\forall x\)
\[
\Rightarrow -f'(-x) = -f'(x) \quad \forall x \quad \text{by chain rule}
\]
\[
\Rightarrow f'(-x) = f'(x) \quad \forall x
\]

Thus \( f'(x) \) is even, as desired.

43 b) False. \( f(x) = x^3 \) is strictly increasing but \( f'(0) \neq 0 \).

b) True. Given \( x_0 \), \( f(x) - f(x_0) > 0 \) \(\forall x < x_0 \), and thus so
\[
x < x_0
\]

are the limits. Thus \( f'(x) \geq 0 \) for all \( x \).

d) True. \( 0 \) is a local minimum for \( f \), so by Lemma 4.16,
\[
f'(0) = 0
\]

d) False. \( f(x) < x \) is a counterexample.