Math221: HW# 7 solutions

Andy Royston

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11.3.13  let \( x = e^{-u} \). Then \( \ln \frac{1}{x} = u \), \( x^2 = e^{-2u} \), and \( dx = -e^{-2u}du \). Furthermore, when \( x = 0 \), \( u \to \infty \), and when \( x = 1 \), \( u = 0 \). Hence

\[
\int_0^1 x^2 \left( \ln \frac{1}{x} \right)^3 \, dx = \int_0^\infty e^{-2u}u^3(-e^{-v}du) = \int_0^\infty e^{-3u}u^3du. \tag{1}
\]

In the last step I used the minus sign to flip the limits of integration. Note that, if we want to apply the definition of the Gamma function to do an integral, it is necessary that the limits of the integral go from 0 to \( \infty \). If they don’t, it’s not a Gamma function. Also, it’s necessary that we have \( e^{-t} \) in the integrand, where \( t \) is the integration variable. In this case, we do not, so we need to make another change of variables. Let \( t = 3u \to du = dt/3 \). Note that the limits do not change:

\[
\int_0^\infty e^{-3u}u^3du = \frac{1}{3^4}\int_0^\infty e^{-t}t^3dt = \frac{1}{3^4}\int_0^\infty e^{-t}t^4-1dt = \frac{1}{3^4}\Gamma(4). \tag{2}
\]

In this case we can simplify the result, using \( \Gamma(n) = (n-1)! \), for \( n \in \mathbb{N} \). Thus

\[
\int_0^1 x^2 \left( \ln \frac{1}{x} \right)^3 \, dx = \frac{3!}{3^4} = \frac{2}{27}. \tag{3}
\]

11.3.16  The initial conditions are \( x(0) = 1, \dot{x}(0) = 0 \). The Lagrangian is

\[
L = T - V = \frac{1}{2}m\ddot{x}^2 - \frac{1}{2}m \ln x. \tag{4}
\]

The Euler-Lagrange equation is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left( m\ddot{x} \right) - \left( -\frac{m}{2x} \right) = 0
\]

\[
\Rightarrow \quad \ddot{x} + \frac{1}{2x} = 0. \tag{5}
\]
This is the second order differential equation that we need to solve. It is of the type \( \frac{d^2x}{dt^2} = F(x) \), with \( F(x) = -\frac{1}{x^2} \). See formulas (7.14) and (7.15), in section 7 of chapter 8. It can be integrated once if we multiply each side by \( \dot{x} \). The result is

\[
\frac{1}{2} \dot{x}^2 = \int F(x)dx = -\int \frac{dx}{2x} + c = -\frac{1}{2} \ln x + c, \quad \text{or} \quad \dot{x}^2 = -\ln x + \dot{c}.
\]  

(6)

This gives us \( \dot{x} \). We still need to integrate again to get \( x(t) \). First, however, use the initial conditions to solve for the integration constant \( \dot{c} \). At time \( t = 0 \) we know that \( x = 1, \dot{x} = 0 \). Plugging these in,

\[
0 = -\ln 1 + \dot{c} = 0 + \dot{c} \quad \Rightarrow \quad \dot{c} = 0.
\]  

(7)

Then

\[
\dot{x}^2 = -\ln x = \ln \frac{1}{x} \quad \Rightarrow \quad \dot{x} = \pm \sqrt{\ln \frac{1}{x}} \rightarrow -\sqrt{\ln \frac{1}{x}}
\]  

\[
\Rightarrow \quad -\frac{dx}{\sqrt{\ln \frac{1}{x}}} = dt.
\]  

(8)

Note that since we know \( x < 1 \), \( \ln x \) is negative, so \( -\ln x = \ln \frac{1}{x} \) is positive. Furthermore, we chose the minus sign from taking the square root because we know that the particle’s velocity is negative; it’s traveling from \( x = 1 \) to the origin. Now, let the time \( T \) be the time it takes for the particle to reach the origin. Then

\[
T = \int_0^T dt = \int_1^0 \frac{-dx}{\sqrt{\ln \frac{1}{x}}} = \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}}.
\]  

(9)

To do the \( x \) integral, make the change of variable \( x = e^{-u} \), so that \( \ln \frac{1}{x} = u \), \( dx = -e^{-u}du \). Also, \( x = 0 \Rightarrow u = \infty \), \( x = 1 \Rightarrow u = 0 \). Then

\[
T = \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \int_0^\infty \frac{-e^{-u}du}{\sqrt{u}} = \int_0^\infty e^{-u}u^{-1/2}du
\]

\[
= \int_0^\infty e^{-u}u^{1/2-1}du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]  

(10)

11.5.4 We want to prove

\[
\Gamma(n + 1/2) = \frac{1 \cdot 3 \cdot \cdots \cdot (2n - 1)}{2^n} \sqrt{\pi} = \frac{(2n)!}{4^n n!} \sqrt{\pi}.
\]  

(11)

Use the fundamental property \( \Gamma(p + 1) = p\Gamma(p) \) for any \( p > 0 \). Then
\[ \Gamma(n + 1/2) = \Gamma((n - 1/2) + 1) = (n - 1/2)\Gamma(n - 1/2). \]  
(12)

Then just keep going:

\[ (n - 1/2)\Gamma(n - 1/2) = (n - 1/2)(n - 3/2)(n - 5/2) \cdots (3/2)(1/2)\Gamma(1/2) \]
(13)

Now, since \( n \) is some fixed integer, this process must come to an end. Eventually, I have to get down to \( \Gamma(1/2) \). Then at this point, I can’t go any further. \( \Gamma(p + 1) = p\Gamma(p) \) only holds for \( p > 0 \). However, I do know that \( \Gamma(1/2) = \sqrt{\pi} \). Thus,

\[ \Gamma(n + 1/2) = \frac{(2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1}{2^n} \sqrt{\pi}, \]  
or
\[ \Gamma(n + 1/2) = \frac{1 \cdot 3 \cdots (2n - 1)}{2^n} \sqrt{\pi}, \]  
(15)

which is the first equality. For the second equality, note that the numerator above is a product of all odd numbers from 1 to \( 2n - 1 \). Let us multiply and divide by the product of all even numbers from 2 to \( 2n \). Then the top will just be the product of all integers from 1 to \( 2n \), otherwise known as \( (2n)! \):

\[ \frac{1 \cdot 3 \cdots (2n - 1)}{2^n} \sqrt{\pi} = \frac{1 \cdot 2 \cdot 3 \cdots (2n - 2)(2n - 1)(2n)}{2^n 2 \cdot 4 \cdots (2n)} \]  
\[ = \frac{(2n)!}{2^n 2 \cdot 4 \cdots (2n)} \sqrt{\pi}. \]  
(16)

Now the product of even numbers from 2 to \( (2n) \) has \( n \) terms. Pull out a 2 from each:

\[ 2 \cdot 4 \cdots (2n) = 2^n(1 \cdot 2 \cdots n) = 2^n n!. \]  
(17)

Hence, noting \( 2^n 2^n = 2^{2n} = (2^2)^n = 4^n \),

\[ \frac{1 \cdot 3 \cdots (2n - 1)}{2^n} \sqrt{\pi} = \frac{(2n)!}{2^n 2^n n!} \sqrt{\pi} = \frac{(2n)!}{4^n n!} \sqrt{\pi}, \]  
(18)

which is the second equality.
11.7.2 We have
\[ \int_0^{\pi/2} \sqrt{\sin^3 x \cos x} dx = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \sin^{3/2} x \cos^{1/2} x dx. \] (19)

Let \( 2p - 1 = 3/2 \Rightarrow p = 5/4 \) and \( 2q - 1 = 1/2 \Rightarrow q = 3/4 \) and compare with formula (6.4) of the text. Then
\[ \frac{1}{2} \cdot 2 \int_0^{\pi/2} \sin^{3/2} x \cos^{1/2} x dx = \frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right). \] (20)

Writing this in terms of Gamma functions,
\[ \frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4} + \frac{3}{4}\right)} = \frac{1}{2} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{2} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \] (21)

where in the last equality I used \( \Gamma(2) = 1! = 1 \). Now there is some trickery that can be done. Write
\[ \Gamma\left(\frac{5}{4}\right) = \Gamma\left(1 + \frac{1}{4}\right) = \frac{1}{4} \Gamma\left(\frac{1}{4}\right). \] (22)

Then
\[ \frac{1}{2} \Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{8} \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right). \] (23)

The reason I did this is because now we have it in the form \( \Gamma(p)\Gamma(1 - p) \), where \( p = 1/4 \). We can use the mystical result, equation (5.4):
\[ \Gamma(p)\Gamma(1 - p) = \frac{\pi}{\sin(\pi p)} \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin(\pi/4)} = \frac{\pi}{1/\sqrt{2}} = \pi \sqrt{2}. \] (24)

Hence, the whole result is
\[ \frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{1}{8} \pi \sqrt{2} \approx 0.55536. \] (25)

11.7.7 Observe that
\[ \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \sin^{2p-1} (\theta) \cos^{2q-1} (\theta) d\theta, \] (26)

if \( p = 1/4, q = 1/2 \). Hence
\[ \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}. \] (27)

I don’t know any way to simplify this further, and neither does Steven Wolfram, so this is the final answer. Numerically, it turns out to be
\[ \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \approx 2.62206. \]  

(28)

### 11.7.9

First we show

\[ B(n, n) = \frac{B(n, \frac{1}{2})}{2^{2n-1}}. \]  

(29)

Start with the trig expression for \( B(n, n) \) and use the trig identity as suggested:

\[
B(n, n) = 2 \int_0^{\pi/2} \sin^{2n-1}(\theta) \cos^{2n-1}(\theta) d\theta = 2 \int_0^{\pi/2} (\sin(\theta) \cos(\theta))^{2n-1} d\theta \\
= 2 \int_0^{\pi/2} \left(\frac{1}{2} \sin(2\theta)\right)^{2n-1} d\theta = \frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin(2\theta))^{2n-1} d\theta. \]  

(30)

Now let’s change variables, \( \phi = 2\theta \). Then \( d\theta = d\phi/2 \), and note that the upper limit changes: when \( \theta = \pi/2 \), \( \phi = \pi \). Thus,

\[
\frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin(2\theta))^{2n-1} d\theta = \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin(\phi))^{2n-1} d\phi = \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin(\phi))^{2n-1}(\cos(\phi)^{2-1/2-1} d\phi. \]  

(31)

Note in the last step that \( (\cos(\phi))^{2-1/2-1} = (\cos(\phi))^0 = 1 \), so I didn’t do anything. Now, this looks like a Beta function except that the upper limit is wrong. But observe that \( \sin(\phi) \) is symmetric about \( \pi/2 \) from 0 to \( \pi \). The area under the graph from 0 to \( \pi/2 \) equals the area under the graph from \( \pi/2 \) to \( \pi \). Hence, this is also true of \( (\sin(\phi))^{2n-1} \). Then the integral from 0 to \( \pi \) is just twice the integral from 0 to \( \pi/2 \). The result follows:

\[
B(n, n) = \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin(\phi))^{2n-1}(\cos(\phi))^{2-1/2-1} d\phi = \frac{1}{2^{2n-1}} \cdot 2 \int_0^{\pi/2} (\sin(\phi))^{2n-1}(\cos(\phi))^{2-1/2-1} d\phi \\
= \frac{1}{2^{2n-1}} B(n, \frac{1}{2}). \]  

(32)

From this result, we can derive the duplication formula for the Gamma function. Simply use the formula

\[
B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} \]  

(33)

on each side of the above result. On the left side,

\[
B(n, n) = \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}. \]  

(34)
On the right side
\[
\frac{1}{2^{2n-1}B(n, \frac{1}{2})} = \frac{1}{2^{2n-1}} \frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n + 1/2)} = \frac{\sqrt{\pi} \Gamma(n)}{2^{2n-1} \Gamma(n + 1/2)}.
\] (35)

Now set them equal and get the result:
\[
\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{\sqrt{\pi} \Gamma(n)}{2^{2n-1} \Gamma(n + 1/2)} \Rightarrow \Gamma(2n) = \frac{2^{2n-1} \Gamma(n)\Gamma(n + 1/2)}{\sqrt{\pi}}.
\] (36)

Finally, let us check this formula for \( n = 1/4 \). On the left side,
\[
\Gamma(2n) \rightarrow \Gamma(2 \cdot \frac{1}{4}) = \Gamma(1/2) = \sqrt{\pi}.
\] (37)

On the right side, we will need the result \( \Gamma(p)\Gamma(1 - p) = \pi / \sin (\pi p) \). We have
\[
\frac{2^{2\cdot1/4-1}\Gamma(1/4)\Gamma(3/4)}{\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\pi}{\sin (\pi/4)} = \frac{1}{\sqrt{2\pi}} \cdot \pi \sqrt{2} = \sqrt{\pi}.
\] (38)

So the formula checks out.

**11.11.4** Use Stirling’s formula in the numerator and denominator. In the limit \( n \rightarrow \infty \), it becomes exact.

\[
\lim_{n \to \infty} \frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2} = \lim_{n \to \infty} \frac{(2n)^{2n}e^{-2n}\sqrt{2\pi(2n)}\sqrt{n}}{2^{2n}(n)^n e^{-n}\sqrt{2\pi n}^2} = \lim_{n \to \infty} \frac{2^{2n}n^{2n}e^{-2n}2n\sqrt{\pi}}{2^{2n}n^{2n}e^{-2n}2\pi n} = \lim_{n \to \infty} \frac{1}{\sqrt{\pi}} = 1.
\] (39)

**11.11.5** We have
\[
= \lim_{n \to \infty} \frac{\Gamma(n + \frac{3}{2})}{\sqrt{n}\Gamma(n + 1)} = \lim_{n \to \infty} \frac{\Gamma((n + 1/2) + 1/2)}{\sqrt{n}\Gamma(n + 1)}
\]
\[
= \lim_{n \to \infty} \frac{(n + 1/2)^{n+1/2}e^{-(n+1/2)}\sqrt{2\pi(n+1/2)}}{\sqrt{n}n^{n}e^{-n}\sqrt{2\pi n}}
\]
\[
= \lim_{n \to \infty} e^{-1/2} \left( \frac{n + 1/2}{n} \right)^{n+1/2+1/2} = e^{-1/2} \lim_{n \to \infty} (1 + \frac{1}{2n})^{n+1}.
\] (40)
For the last limit, put it in a more standard form:

$$
\lim_{n \to \infty} (1 + \frac{1}{2n})^{n+1} = \lim_{n \to \infty} (1 + \frac{1}{2n})^n \lim_{n \to \infty} (1 + \frac{1}{2n}) = \lim_{n \to \infty} (1 + \frac{1}{2n})^n
$$

$$
= \lim_{n \to \infty} \sqrt[n]{(1 + \frac{1}{2n})^{2n}} = \sqrt[n]{\lim_{n \to \infty} (1 + \frac{1}{2n})^{2n}}. \quad (41)
$$

Here I pulled off one factor of \((1 + \frac{1}{2n})\), and took its limit, which is 1. Then I introduced the square root so that the remaining limit has the form

$$
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x, \quad (42)
$$

(just let \(x = 2n\)). This limit is something we all supposedly learned in high school.... The answer is \(e\). If you just quote this or look it up, that’s fine by me, but it’s actually pretty easy to prove. The trick is write it as an exponential of a log:

$$
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} e^{\ln ((1+1/x)^x)} = \lim_{x \to \infty} e^{x \ln (1+1/x)}. \quad (43)
$$

Now bring the limit in the exponential; you can do this–bring the limit inside the function–with any continuous function. (That’s what it really means to be continuous). Then observe that, using the Taylor series \(\ln (1 + \epsilon) = \epsilon + O(\epsilon^2)\), for \(\epsilon = 1/x\) small,

$$
\lim_{x \to \infty} x \ln (1 + 1/x) = \lim_{x \to \infty} x(1/x + O((1/x)^2)) = \lim_{x \to \infty} (1 + O(1/x)) = 1. \quad (44)
$$

Therefore, the thing in the exponential goes to 1, and we get

$$
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e. \quad (45)
$$

Plugging this result into our problem,

$$
\lim_{n \to \infty} \frac{\Gamma(n + \frac{3}{2})}{\sqrt{n!\Gamma(n + 1)}} = e^{-1/2} \lim_{n \to \infty} (1 + \frac{1}{2n})^{n+1} = e^{-1/2} \sqrt{e} = 1. \quad (46)
$$

12.1.2 Let’s first solve the equation by pedestrian methods:

$$
y' = 3x^2 y \quad \Rightarrow \quad \frac{dy}{y} = 3x^2 dx \quad \Rightarrow \quad \ln y = x^3 + c
$$

$$
\Rightarrow \quad y(x) = Ae^{x^3}. \quad (47)
$$
Now for the power series solution, assume

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]  

(48)

for some coefficients \( a_n \) that we need to solve for. Then

\[ 3x^2 y = 3a_0 x^2 + 3a_1 x^3 + 3a_2 x^4 + \cdots + 3a_{n-2} x^n + \cdots. \]  

(49)

The derivative is

\[ y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n + 1) a_{n+1} x^n + \cdots. \]  

(50)

Now write out \( y' - 3x^2 y = 0 \), grouping powers of \( x \).

\[ y' - 3x^2 y = a_1 + 2a_2 x + (3a_3 - 3a_0) x^2 + (4a_4 - 3a_1) x^3 + \cdots + ((n + 1) a_{n+1} - 3a_{n-2}) x^n + \cdots = 0. \]  

(51)

Now here’s why we did this. The above equation has to hold for all \( x \). (It’s just the original differential equation, with our guess solution plugged in). Since the \( x^n \) are linearly independent functions for each \( n = 0, 1, 2, \ldots \), the only way the whole thing will be zero is if the coefficient of every power of \( x \) is zero. Thus, we get a list of equations:

\[ a_1 = 0, \]
\[ 2a_2 = 0, \]
\[ 3(a_3 - a_0) = 0, \]
\[ (4a_4 - 3a_1) = 0, \]
\[ (5a_5 - 3a_2) = 0, \]
\[ (6a_6 - 3a_3) = 0, \]
\[ \ldots \]
\[ ((n + 1) a_{n+1} - 3a_{n-2}) = 0, \]  

(52)

\[ \ldots \]

In particular, we learn that \( a_1 = a_2 = 0 \), but \( a_3 = a_0 \). Then \( a_4 \propto a_1 = 0 \) and \( a_5 \propto a_2 = 0 \), but \( a_6 = \frac{1}{2} a_3 = \frac{1}{2} a_0 \). This pattern continues; the general term tells us that a coefficient \( a_n \) is always connected to three back, \( a_{n-3} \). We will have \( a_7 = a_8 = 0 \), but

\[ 9a_9 = 3a_6 \quad \Rightarrow \quad a_9 = \frac{1}{3} a_6 = \frac{1}{3 \cdot 2} a_3 = \frac{1}{3 \cdot 2 \cdot 1} a_0. \]  

(53)

You can now guess the pattern:
\[ a_{3m} = \frac{1}{m(m-1) \cdots \cdot 2 \cdot 1} a_0, \quad a_{3m-1} = a_{3m-2} = 0, \quad m \in \mathbb{N}. \quad (54) \]

Thus we have determined all of the coefficients in terms of a single undetermined one, \( a_0 \). This is expected. We have one undetermined coefficient because we started with a first order equation. Let us now write down the solution:

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{m=1}^{\infty} a_{3m} x^{3m} = a_0 + a_0 \sum_{m=1}^{\infty} \frac{1}{m!} x^{3m} \\
= a_0 \sum_{m=0}^{\infty} \frac{1}{m!} (x^3)^m = a_0 e^{x^3}. \quad (55)\]

Hence we get the correct result, with \( a_0 \) serving as \( A \), the arbitrary constant.

**12.1.7** The differential equation is \( x^2 y'' - 3xy' + 3y = 0 \). This is a Cauchy type equation. To get the solution, try \( y = x^k \). Then

\[
x^2 k(k-1)x^{k-2} - 3xk x^{k-1} + 3x^k = (k(k-1) - 3k + 3)x^k = 0 \\
\Rightarrow k^2 - 4k + 3 = (k-3)(k-1) = 0 \quad \Rightarrow k = 1, 3. \quad (56)\]

Therefore the general solution is

\[ y(x) = Ax + Bx^3. \quad (57) \]

Let’s try it by power series.

\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots, \\
y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots. \quad (58)\]

Then

\[
x^2 y'' = 2a_2 x^2 + 3 \cdot 2a_3 x^3 + 4 \cdot 3a_4 x^4 + \cdots + n(n-1)a_n x^n + \cdots, \\
-3xy' = -3a_1 x + (-3)2a_2 x^2 + (-3)3a_3 x^3 + \cdots + (-3)na_n x^n + \cdots. \quad (59)\]

Now write down the differential equation term by term in \( x \):

\[
x^2 y'' - 3xy' + 3y = 3a_0 + (-3 + 3)a_1 x + (2 \cdot 1 - 3 \cdot 2 + 3)a_2 x^2 + (3 \cdot 2 - 3 \cdot 3 + 3)a_3 x^3 + \cdots \\
+ (n(n-1) - 3n + 3)a_n x^n + \cdots = 0 \quad (60)\]
Every coefficient must vanish. Look at the general term, which holds for all \( n \):

\[
(n(n - 1) - 3n + 3)a_n = (n - 3)(n - 1)a_n = 0, \quad n = 0, 1, 2, \ldots
\]  

(61)

If \( n = 1, 3 \) then this is satisfied, and there are no restrictions on \( a_1, a_3 \). However, for all other \( n \), we must require \( a_n = 0 \). Hence, the solution is

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n = a_1 x + a_3 x^3.
\]

(62)

This is the solution we had before with \( a_1 = A, a_3 = B \).

12.1.8 The equation is \((x^2 + 2x)y'' - 2(x + 1)y' + 2y = 0\). I actually don’t see a way to solve this, except by power series. So let’s find the power series solution first and then check the solutions we get. \( y, y' \), and \( y'' \) are as in equation (58) above. All the terms we need are

\[
x^2 y'' = 2 \cdot 1a_2 x^2 + 3 \cdot 2a_3 x^3 + 4 \cdot 3a_4 x^4 + \cdots,
\]

\[
2xy'' = (2)2 \cdot 1a_2 x + (2)3 \cdot 2a_3 x^2 + (2)4 \cdot 3a_4 x^3 + \cdots,
\]

\[
-2xy' = (-2)a_1 x + (-2)2a_2 x^2 + (-2)3a_3 x^3 + (-2)4a_4 x^4 + \cdots,
\]

\[
-2y' = (-2)a_1 + (-2)2a_2 x + (-2)3a_3 x^2 + (-2)4a_4 x^3 + \cdots,
\]

\[
2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots.
\]

(63)

Adding everything up, grouping in powers of \( x \),

\[
(-2a_1 + 2a_0) + ((2)2 \cdot 1a_2 - 2(a_1 + 2a_2) + 2a_1)x
\]

\[
+ (2 \cdot 1a_2 + (2)3 \cdot 2a_3 - 2(2a_2 + 3a_3) + 2a_2)x^2 + (3 \cdot 2a_3 + (2)4 \cdot 3a_4 - 2(3a_3 + 4a_4) + 2a_3)x^3
\]

\[
+ \cdots + (n(n - 1)a_n + (2)(n + 1)na_{n+1} - 2(na_n + (n + 1)a_{n+1}) + 2a_n)x^n + \cdots = 0.
\]

(64)

The general term actually gives the correct form for all \( n = 0, 1, 2, \ldots \). Hence, we require

\[
(n(n - 1)a_n + (2)(n + 1)na_{n+1} - 2(na_n + (n + 1)a_{n+1}) + 2a_n) = 0, \quad \forall n = 0, 1, \ldots
\]

\[
\Rightarrow \quad (n(n - 1) - 2n + 2)a_n + (2n(n + 1) - 2(n + 1))a_{n+1} = 0
\]

\[
\Rightarrow \quad (n^2 - 3n + 2)a_n + 2(n + 1)(n - 1)a_{n+1} = 0
\]

\[
\Rightarrow \quad (n - 2)(n - 1)a_n + 2(n + 1)(n - 1)a_{n+1} = 0.
\]

(65)

Now, as long as \( n \neq 1 \), we can divide out the \((n - 1)\) and obtain

\[
a_{n+1} = \frac{-(n - 2)}{2(n + 1)}a_n, \quad n \neq 1.
\]

(66)
For \( n = 0 \) we have \( a_0 = a_1 \). For \( n = 1 \), we learn nothing about the \( a \)'s. Equation (65) is automatically satisfied. Now, for \( n = 2 \), the equation says \( a_3 = 0a_2 = 0 \). Then, from this, 

\[
a_4 = \frac{-1}{2 \cdot 4} 0 = 0, \quad a_5 \propto a_4 = 0, \quad \ldots.
\]

That is, all of the \( a_n \), starting with \( a_3 \) are 0. Note that we never learn anything about \( a_2 \). Summarizing,

\[
a_0 = a_1 = \text{arbitrary const.} \\
\quad a_2 = \text{arbitrary const.} \\
\quad a_n = 0, \quad n \geq 3.
\]

Hence, \( y(x) \) is given by

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0(1 + x) + a_2 x^2,
\]

where \( a_0, a_2 \) are arbitrary constants. You can plug in each of these functions, \((1 + x)\) and \(x^2\), and verify that they are indeed solutions to the differential equation.

**12.2.1** Legendre’s equation is \((1 - x^2)y'' - 2xy' + l(l+1)y = 0\). The general series solution is

\[
y = a_0 \left[ 1 - \frac{l(l+1)}{2!} x^2 + \frac{l(l+1)(l-2)(l+3)}{4!} x^4 - + \ldots \right] \\
+ a_1 \left[ x - \frac{(l-1)(l+2)}{3!} x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!} x^5 - + \ldots \right].
\]

When \( l \) is an integer, one of these series will terminate after a finite number of terms. These special solutions are the Legendre polynomials \( P_l(x) \). To find them, just see which series is going to stop, and write down all the nonzero terms. For \( P_2(x) \), look at the top series. At the third term, and in all subsequent terms, there is the factor \((l-2)\), which will vanish. Therefore we are interested in the solution

\[
y(x) = a_0 \left( 1 - \frac{2(2+1)}{2!} x^2 \right) = a_0(1 - 3x^2).
\]

In order to make it the Legendre polynomial \( P_2(x) \), we fix the constant \( a_0 \) by demanding \( P_2(1) = 1 \). This gives

\[
1 = a_0(1 - 3) \quad \Rightarrow \quad a_0 = \frac{1}{2}.
\]

Hence,
\[ P_2(x) = \frac{1}{2}(3x^2 - 1). \]  
\hspace{1cm} (73)

For the next one, \( l = 3 \), look at the lower series. The third term, and all terms afterwards, have the factor \((l - 3)\). Therefore, we are interested in
\[ y(x) = a_1 \left( x - \frac{(3 - 1)(3 + 2)}{3!} x^3 \right) = a_1 \left( x - \frac{5}{3} x^3 \right). \]  
\hspace{1cm} (74)

Normalize by \( P_3(1) = 1 \):
\[ 1 = a_1 \left( 1 - \frac{5}{3} \right) = -a_1 \frac{2}{3} \quad \Rightarrow \quad a_1 = -\frac{3}{2}. \]  
\hspace{1cm} (75)

Thus
\[ P_3(x) = \frac{3}{2} \left( \frac{5}{3} x^3 - x \right) = \frac{1}{2} (5x^3 - 3x). \]  
\hspace{1cm} (76)

Finally, for \( l = 4 \), the top series will terminate at the fourth term. The fourth term is given by
\[ \frac{l(l + 1)(l - 2)(l + 3)(l - 4)(l + 5)}{6!} x^6. \]  
\hspace{1cm} (77)

You can see this by the recursion relation, formula (2.6) of the text, or just observe the pattern. The point is that it, and every term after it, contain the factor \((l - 4)\), and so vanish. So write
\[ y(x) = a_0 \left( 1 - \frac{4(4 + 1)}{2!} x^2 - \frac{4(4 + 1)(4 - 2)(4 + 3)}{4!} x^4 \right) = a_0 \left( 1 - 10x^2 + \frac{35}{3} x^4 \right). \]  
\hspace{1cm} (78)

Normalizing,
\[ 1 = a_0 \left( 1 - 10 + \frac{35}{3} \right) = a_0 \frac{3}{3} (3 - 30 + 35) = \frac{8}{3} a_0 \quad \Rightarrow \quad a_0 = \frac{3}{8}. \]  
\hspace{1cm} (79)

Hence,
\[ P_4(x) = \frac{3}{8} \left( 1 - 10x^2 + \frac{35}{3} x^4 \right) = \frac{1}{8} (3 - 30x^2 + 35x^4). \]  
\hspace{1cm} (80)
12.2.2 Observe that in the previous problem, we always used the top series for \( l \) even and the bottom series for \( l \) odd. This will always be the case because the general term in the top series has structure

\[
l(l-2)(l-4)\cdots(l+1)(l+3)\cdots \tag{81}
\]

while the general term in the bottom series has structure

\[
(l-1)(l-3)(l-5)\cdots(l+2)(l+4)\cdots \tag{82}
\]

So if \( l \) is even, (and remember, the Legendre polynomials are only defined for \( l \geq 0 \)), then the top series will eventually terminate, while the bottom does not. On the other hand, if \( l \) is odd, the bottom series will be the one that terminates.

Now, the top series contains only even powers of \( x \), so it is an even function. The bottom series contains only odd powers of \( x \), so it is an odd function. Therefore we learn in general

\[
P_l(x) = P_l(-x), \quad l \text{ even};
\]

\[
P_l(x) = -P_l(-x), \quad l \text{ odd}. \tag{83}
\]

Now apply this result to \( x = 1 \). There we know, (by definition really), that \( P_l(1) = 1 \), for any \( l \), even or odd. Hence,

\[
1 = P_l(1) = P_l(-1), \quad l \text{ even};
\]

\[
1 = P_l(1) = -P_l(-1), \quad l \text{ odd}. \tag{84}
\]

This may be summarized as

\[
P_l(-1) = (-1)^l. \tag{85}
\]