8.11.6

a  Let $y_n(t)$ be the function which satisfies $Ay''_n + By'_n + Cy_n = f_n(t)$, $y_{n0} = 0$, and $y'_{n0} = 0$, where the forcing term is given by

$$f_n(t) = \begin{cases} 
0, & t < t_0 \\
n, & t_0 < t < t_0 + 1/n \\
0, & t_0 + 1/n < t
\end{cases}$$  \hspace{1cm} (1)

for $n \in \mathbb{N}$. Taking the Laplace transform of both sides gives

$$(Ap^2 + Bp + C)Y_n = L[f_n(t)] \Rightarrow Y_n = \frac{1}{Ap^2 + Bp + C}L[f_n(t)]$$

$$= \frac{1}{A(p + a)(p + b)}L[f_n(t)].$$  \hspace{1cm} (2)

In the transfer function, I’ve implicitly defined the constants $a$ and $b$ in terms of the $A, B, C$. The exact relation can easily be derived by demanding the equality $Ap^2 + Bp + C = A(p + a)(p + b)$, but it won’t really be necessary here. (See the next problem for an actual computation of the $a, b$, given the $A, B, C$). The transfer function is the same for each $n$, and its inverse transform is

$$L^{-1}\left[\frac{1}{A(p + a)(p + b)}\right] = \frac{e^{-at} - e^{-bt}}{A(b - a)}. \hspace{1cm} (3)$$

Then we find, by convolution,

$$y_n(t) = \int_0^t \frac{e^{-a(t-\tau)} - e^{-b(t-\tau)}}{A(b - a)}f_n(\tau)d\tau. \hspace{1cm} (4)$$

To do the integral, there are three cases that need to be considered separately. First, if $t < t_0$, $f_n(\tau)$ is zero over the entire interval of integration, $(0, t)$. Hence, the integral is 0. Second, consider $t_0 < t < t_0 + 1/n$. Break the integral into two parts: $\int_0^t = \int_0^{t_0} + \int_0^{t_0}$. The integral over $(0, t_0)$ does not contribute because $f(\tau)$ is zero there. While over $(t_0, t)$, $f(\tau) = n$. Hence,
\[ y_n(t) = \frac{n}{A(b-a)} \int_{t_0}^{t} (e^{-a(t-\tau)} - e^{-b(t-\tau)}) d\tau = \frac{n}{A(b-a)} \left( \frac{e^{-at}}{a} (e^{at} - e^{at_0}) - \frac{e^{-bt}}{b} (e^{bt} - e^{bt_0}) \right) \]

\[ = \frac{n}{A(b-a)} \left( \frac{1 - e^{-a(t-t_0)}}{a} - \frac{1 - e^{-b(t-t_0)}}{b} \right), \quad t_0 < t < t_0 + \frac{1}{n}. \]  

Finally, if \( t > t_0 + 1/n \), break the integral into three pieces: \( \int_{t_0}^{t} = \int_{0}^{t_0} + \int_{t_0}^{(t_0+1)/n} + \int_{(t_0+1)/n}^{t} \). This time the first and third integrals don’t contribute because \( f(\tau) \) is zero over those intervals. Thus,

\[ y_n(t) = \frac{n}{A(b-a)} \int_{t_0}^{t_0+1/n} (e^{-a(t-\tau)} - e^{-b(t-\tau)}) d\tau \]

\[ = \frac{n}{A(b-a)} \left( \frac{e^{-at}}{a} (e^{a(t_0+1/n) - e^{at_0})} - \frac{e^{-bt}}{b} (e^{b(t_0+1/n) - e^{bt_0})} \right) \]

\[ = \frac{n}{A(b-a)} \left( \frac{1}{a} (e^{a/n} - 1) - \frac{1}{b} (e^{b/n} - 1) \right), \quad t > t_0 + \frac{1}{n}. \]  

In summary,

\[ y_n(t) = \frac{n}{A(b-a)} \cdot \begin{cases} 
0, & t < t_0 \\
\frac{1 - e^{-a(t-t_0)}}{a} - \frac{1 - e^{-b(t-t_0)}}{b}, & t_0 < t < t_0 + 1/n \\
\frac{e^{-a(t-t_0)}}{a} (e^{a/n} - 1) - \frac{e^{-b(t-t_0)}}{b} (e^{b/n} - 1), & t_0 + 1/n < t
\end{cases} \]  

Now we want to take the limit \( n \to \infty \). Let \( y(t) = \lim_{n \to \infty} (y_n(t)) \). Clearly \( y = 0 \) for \( t < t_0 \). We can ignore the middle interval since this shrinks to zero length. The third interval becomes \( (t_0, \infty) \) in the limit. In this interval,

\[ y(t) = \lim_{n \to \infty} \frac{n}{A(b-a)} \left( \frac{e^{-at}}{a} (e^{a/n} - 1) - \frac{e^{-bt}}{b} (e^{b/n} - 1) \right) \]

\[ = \frac{1}{A(b-a)} \left( \frac{e^{-at}}{a} \lim_{n \to \infty} n(e^{a/n} - 1) - \frac{e^{-bt}}{b} \lim_{n \to \infty} n(e^{b/n} - 1) \right). \]  

We need to evaluate limits of the form \( n(e^{c/n} - 1) \) as \( n \to \infty \). One way is to let \( x = 1/n \) and look at the limit as \( x \to 0 \); ie.

\[ \lim_{n \to \infty} n(e^{c/n} - 1) = \lim_{x \to 0} \frac{e^{cx} - 1}{x}. \]  

This can easily be evaluated by Taylor expanding the numerator:

\[ \lim_{x \to 0} \frac{e^{cx} - 1}{x} = \lim_{x \to \infty} \frac{1 + cx + O(x^2)) - 1}{x} = \lim_{x \to \infty} \frac{cx + O(x^2)}{x} = \lim_{x \to \infty} (c + O(x)) = c. \]  

\[ \]
(You could also use L’Hospital’s rule to evaluate the limit). Plugging in the result with \(c = a\) and \(c = b\), we find

\[
y(t) = \frac{1}{A(b-a)}(e^{-a(t-t_0)} - e^{-b(t-t_0)}), \quad t > t_0.
\]  

(11)

In summary,

\[
y(t) = \lim_{n \to \infty} y_n(t) = \begin{cases} 
0, & t < t_0 \\
\frac{1}{A(b-a)}(e^{-a(t-t_0)} - e^{-b(t-t_0)}), & t_0 < t 
\end{cases}.
\]

(12)

b  Now we find the solution to \(Ay'' + By' + Cy = \delta(t - t_0)\), \(y_0 = y_0' = 0\). It should give the same result as the limiting process above. Again, use convolution:

\[
Y = \frac{1}{A(p+a)(p+b)}L[\delta(t - t_0)] \implies y(t) = \int_0^t \frac{e^{-a(t-\tau)} - e^{-b(t-\tau)}}{A(b-a)}\delta(\tau - t_0)d\tau.
\]

(13)

Note the form of this. If \(f(t)\) is the inverse transform of the transfer function, and \(g(t) = \delta(t - t_0)\), then I am writing the convolution as \(\int_0^t f(t-\tau)g(\tau)\). Whenever there is a function defined piecewise, or a special function like the delta function, it’s probably better let the argument of that function be \(\tau\), and the argument of the other function being convolved with, \(t - \tau\). You can do it the other way, of course, but you have to be very careful about how the step function, etc. is affected.

Now, there are two cases. If \(t < t_0\), then the zero of the \(\delta\) function, \(t = t_0\), is not in the interval of integration. Hence, the result is zero. If \(t > t_0\), then it is, and we get the inverse transform of the transfer function evaluated at \(t_0\):

\[
y(t) = \frac{1}{A(b-a)} \int_0^t (e^{-a(t-\tau)} - e^{-b(t-\tau)})\delta(\tau - t_0)d\tau
\]

\[
= \begin{cases} 
0, & t < t_0 \\
\frac{1}{A(b-a)}(e^{-a(t-t_0)} - e^{-b(t-t_0)}), & t_0 < t 
\end{cases}.
\]

(14)

Indeed, this is the same as the limit in part (a).

c  Letting \(t_0 = 0\), we have

\[
y(t) = \begin{cases} 
0, & t < 0 \\
\frac{1}{A(b-a)}(e^{-at} - e^{-bt}), & t > 0 
\end{cases}.
\]

(15)

On the other hand, the inverse Laplace transform of the transfer function is, using \(L7\),

\[
L^{-1}\left[\frac{1}{A(p+a)(p+b)}\right] = \frac{1}{A} L^{-1}\left[\frac{1}{(p+a)(p+b)}\right] = \frac{e^{-at} - e^{-bt}}{A(b-a)}.
\]

(16)
But, remember, that the Laplace transform/inverse transform of a function only cares about the function on \((0, \infty)\), so we may always define it to be zero on \((-\infty, 0)\). Then we see that \(y\) is the inverse transform of the transfer function.

**8.11.9** We solve \(y'' + 2y' + 10y = \delta(t - t_0)\), using formula (14). Obviously \(A = 1\). For \(a, b\), let us find the roots of the polynomial \(p^2 + 2p + 10 = 0\). They are

\[
p_{\pm} = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i. \tag{17}\]

These are not \(a, b\), but we can easily see how they are related:

\[
A(p + a)(p + b) = (p + a)(p + b) = p^2 + 2p + 10 = (p - p_+)(p - p_-). \tag{18}\]

So we may set \(a = -p_+\) and \(b = -p_-\), or

\[
a = 1 - 3i, \quad b = 1 + 3i. \tag{19}\]

Then

\[
y(t) = e^{-(1-3i)(t-t_0)} - e^{-(1+3i)(t-t_0)} \frac{e^{-3i(t-t_0)} - e^{-3i(t-t_0)}}{(1+3i) - (1-3i)} = \frac{1}{3} e^{-(t-t_0)} \sin (3(t-t_0)), \quad t > t_0. \tag{20}\]

Then

\[
y(t) = \begin{cases} 0, & t < t_0 \\ \frac{1}{3} e^{-(t-t_0)} \sin (3(t-t_0)), & t > t_0 \end{cases}. \tag{21}\]

**8.11.12** The functions \(f_n(x - a)\) are given by

\[
f_n(x - a) = \frac{1}{2\pi} \int_{-n}^{n} e^{i\alpha(x-a)} d\alpha = \frac{1}{2\pi} \cdot \frac{e^{i\alpha(x-a)} \bigg|_{-n}^{n}}{i(x-a)} = \frac{1}{2\pi i(x-a)}(e^{in(x-a)} - e^{-in(x-a)}) = \frac{1}{\pi(x-a)} \sin (n(x - a)) \bigg|_{-n}^{n} = \frac{n \sin (n(x - a))}{\pi n(x-a)}. \tag{22}\]

See Figure 1, for plots of several different \(n\) values. Now consider the integral of \(f_n\) over \(\mathbb{R}\):

\[
\int_{-\infty}^{\infty} f_n(x - a) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin (n(x - a))}{n(x-a)} n dx. \tag{23}\]
Figure 1: Problem 8.11.12. The plots are color-coated by their increasing $n$ values in the order red, green, blue. In particular, we see that as $n$ grows, the peak at $x = a$ becomes higher and higher, while the oscillations die off more quickly.

Let $\theta = n(x - a)$. Then $ndx = d\theta$, and the integral still goes from $-\infty$ to $\infty$. Hence,

$$\int_{\mathbb{R}} f_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \theta}{\theta} d\theta.$$  \hspace{1cm} (24)

Note that the dependence on $n$ has already disappeared. The integral we have left to do is well known. There are several tricks to evaluate it, but I don’t know any that are sufficiently simple for me to present here. I will just quote the result:

$$\int_{-\infty}^{\infty} \frac{\sin \theta}{\theta} d\theta = \pi.$$  \hspace{1cm} (25)

Hence,

$$\int_{-\infty}^{\infty} f_n(x - a)dx = 1.$$  \hspace{1cm} (26)

8.12.6 We want to find the Green function for the differential operator $D^2 - a^2$, satisfying the initial conditions $G = 0$ and $dG/dt = 0$ at $t = 0$. In other words, we want to find the particular solution of $(D^2 - a^2)G = \delta(t - t')$. This is most easily solved by convolution. We have

$$L[G] = \frac{1}{p^2 - a^2} L[\delta(t - t')]$$  \hspace{1cm} (27)

The inverse transform of the transfer function is $\frac{1}{a} \sinh(at)$; (L9). Thus,

$$G = \frac{1}{a} \int_0^t \sinh(a(t - \tau))\delta(\tau - t').$$  \hspace{1cm} (28)
The usual arguments apply. If \( t < t' \) then the interval of integration does not include the zero of the \( \delta \) function; hence, the result is zero. If \( t > t' \), then the interval does contain the zero of the \( \delta \) function, and we simply replace \( \tau \) with \( t' \) in the argument of the sinh. Thus,

\[
G(t, t') = \begin{cases} 
0, & t < t' \\
\frac{1}{a} \sinh (a(t - t')), & t' < t
\end{cases}
\]  

(Note that it follows from the expression (28) that \( G = 0 \) if either \( t \) or \( t' \) is less than zero. If \( t < 0 \), then we are integrating over the empty set; there are no points in \((0, t)\). If \( t' < 0 \), then the zero of the \( \delta \) function is never in the range of integration, so the result is again zero).

Now we use the Green function to solve \((D^2 - a^2)y = f(t)\), where \( f(t) = 1 \) on \((0, \infty)\). We have

\[
y(t) = \int_0^\infty G(t, t') f(t') dt' = \int_0^t \frac{1}{a} \sinh (a(t - t'))(1) dt' + \int_t^\infty 0
ty(t) = -\frac{1}{a^2} \cosh (a(t - t'))\bigg|_0^t = \frac{1}{a^2} (\cosh (at) - 1).
\]

Here we used the fact that the Green function is zero for \( t < t' \).

**8.12.7** We use the Green function found in the previous problem to solve \( y'' - a^2 y = e^{-t}, \ y_0 = 0 = y'_0 \). According to equation (12.4) of the text and equation (29) above,

\[
y(t) = \int_0^\infty G(t, t') e^{-t'} dt' = \int_0^t \left( \frac{1}{a} \sinh (a(t - t')) \right) e^{-t'} dt' + \int_t^\infty 0 dt'.
\]

Writing out the sinh,

\[
y(t) = \frac{1}{2a} \int_0^t (e^{a(t-t')} - e^{-a(t-t')}) e^{-t'} dt' = \frac{1}{2a} \int_0^t (e^{at} e^{-(a+1)t} - e^{-at} e^{(a-1)t}) dt'
y(t) = \frac{1}{2a} \left( \frac{e^{at}}{-(a+1)} - \frac{e^{-at}}{a-1} \right) = \frac{1}{2a} \left( \frac{(a-1)(e^{-t} - e^{at}) + (a+1)(e^{-t} - e^{-at})}{-(a+1)(a-1)} \right)
y(t) = \frac{1}{(2a)(1-a^2)} \left( a(e^{-t} - e^{at} + e^{-t} - e^{-at}) + (e^{-t} - e^{-at} - e^{-t} + e^{at}) \right)
y(t) = \frac{1}{(2a)(1-a^2)} \left( 2a(e^{-t} - \cosh (at)) + 2 \sinh (at) \right)
y(t) = \frac{1}{a(a^2-1)} \left( a(\cosh (at) - e^{-t}) - \sinh (at) \right).
\]
9.2.3 We see that \( F(x, y, y') = x\sqrt{1 - y'^2} \). Since \( \partial F / \partial y = 0 \), we have

\[
\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0 \quad \Rightarrow \quad \frac{d}{dx} \left( \frac{-xy'}{\sqrt{1 - y'^2}} \right) = 0.
\]

(33)

Now, all of the stuff in parentheses is some unknown function of \( x \); call it \( f(x) \). The equation says that \( \frac{df}{dx} = 0 \). It follows that \( f(x) \) is a constant, say \( c \). Hence, the Euler equation implies

\[
\frac{-xy'}{\sqrt{1 - y'^2}} = c.
\]

(34)

This is now a first order differential equation for \( y(x) \), (which is what we are trying to solve for). Untangle \( y'(x) \) and then we will see it is separable:

\[
\frac{-xy'}{\sqrt{1 - y'^2}} = c \quad \Rightarrow \quad x^2y'^2 = c^2(1 - y'^2) \quad \Rightarrow \quad y'^2(x^2 + c^2) = c^2
\]

\[
\Rightarrow \quad y' = \frac{c}{\sqrt{x^2 + c^2}} \quad \Rightarrow \quad dy = \frac{cdx}{\sqrt{x^2 + c^2}}.
\]

(35)

You can look up the integral of the right side. It is done by hyperbolic trig substitution. In particular, take \( x = c\sinh \xi \). The reason is because the hyperbolic trig functions satisfy \( \cosh^2 - \sinh^2 = 1 \), so \( 1 + \sinh^2 = \cosh^2 \), which is what we will have in the denominator. Also, \( dx = c\cosh \xi d\xi \). Thus,

\[
\int \frac{cdx}{\sqrt{c^2 + x^2}} = \int \frac{c^2 \cosh \xi d\xi}{\sqrt{c^2 + c^2 \sinh^2 \xi}} = \int \frac{c^2 \cosh \xi d\xi}{c \cosh \xi} = \int cd\xi = c\xi + b,
\]

(36)

where \( b \) is another integration constant. Now, if \( x = c\sinh \xi \), then \( \xi = \text{arcsinh}(x/c) \). Hence,

\[
y = c \text{arcsinh}(x/c) + b \quad \text{or} \quad \sinh \left( \frac{y}{c} - b \right) = \frac{x}{c}.
\]

(37)

Defining \( A = 1/c \) and \( B = -b \), I get the book result

\[
\sinh (Ay + B) = Ax.
\]

(38)

9.2.5 This time \( F(x, y, y') = y'^2 + y^2 \). Therefore, the Euler equation gives

\[
\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = \frac{d}{dx} (2y') - 2y = 0
\]

\[
\Rightarrow \quad y'' - y = 0.
\]

(39)
This is a homogeneous, linear 2nd order equation with constant coefficients. The solution is $Ae^{x} + Be^{-x}$, where $r_\pm$ are the roots of $r^2 - 1 = 0$. So $r_\pm = \pm 1$, and the general solution is

$$y(x) = Ae^{x} + Be^{-x} = a \cosh x + b \sinh x.$$ (40)

**9.3.4** Since $x$ does not appear in $F$, let’s make $x$ the dependent variable; ie $x = x(y)$, $x' = dx/dy$. We have the result from calculus, $dx/dy = (dy/dx)^{-1}$, or $x' = 1/y'$. Also note $dx = dx/dy dy = x' dy$.

Hence,

$$\int y \sqrt{y'^2 + y^2} dx = \int y \sqrt{1/x'^2 + y^2} dy = \int y \sqrt{1 + x'^2y^2} dy.$$ (41)

Therefore, the integral will be stationary if $\tilde{F}(y, x, x') = y \sqrt{1 + x'^2y^2}$ satisfies the Euler equation

$$\frac{d}{dy} \left( \frac{\partial \tilde{F}}{\partial x'} \right) - \frac{\partial \tilde{F}}{\partial x} = 0 \Rightarrow \frac{d}{dy} \left( \frac{x'y}{\sqrt{1 + x'^2y^2}} \right) = 0.$$ (42)

Hence, (by the same arguments as in problem 9.2.3), we obtain a first integral,

$$\frac{x'y^3}{\sqrt{1 + x'^2y^2}} = c,$$ (43)

for some constant $c$.

**9.3.6** Again, $x$ does not appear in $F(x, y, y') = yy'^2/(1 + yy')$, so let’s change the dependent variable to $x$.

$$\int \frac{yy'^2}{1 + yy'} dx = \int \frac{y(1/x')^2}{1 + y(1/x')} x' dy = \int \frac{y(1/x')}{1 + y(1/x')} dy = \int \frac{y}{x' + y} dy.$$ (44)

The integral is stationary if $\tilde{F}(y, x, x') = y/(x' + y)$ satisfies the Euler equation

$$\frac{d}{dy} \left( \frac{\partial \tilde{F}}{\partial x'} \right) - \frac{\partial \tilde{F}}{\partial x} = 0 \Rightarrow \frac{d}{dy} \left( \frac{-y}{(x' + y)^2} \right) = 0.$$ (45)

We obtain the first integral

$$\frac{y}{(x' + y)^2} = c.$$ (46)

(I absorbed the minus sign into the arbitrary constant $c$). This is a first order differential equation for $x(y)$. Solve for $x'$ as a function of $y$ and integrate:

$$\frac{y}{(x' + y)^2} = c \Rightarrow \sqrt{y/c} = x' + y \Rightarrow x' = \sqrt{y/c} - y$$

$$\Rightarrow x = \int (\sqrt{y/c} - y) dy = \frac{2}{3} y^{3/2}/\sqrt{c} - \frac{1}{2} y^2 + b,$$ (47)
or, redefining the constant $c$,

$$x = ay^{3/2} - \frac{1}{2}y^2 + b.$$  

(48)