1 Exercises

**Exercise 1.1.** If $F$ is a field, then either $\mathbb{Q} \subseteq F$ or $\mathbb{F}_p \subseteq F$ for some prime $p$ (i.e. $\mathbb{Q} \hookrightarrow F$, $\mathbb{F}_p \hookrightarrow F$).

**Exercise 1.2.** Find a $2 \times 2$ matrix $M$ such that there is no basis for $\mathbb{R}^2$ consisting of eigenvectors of $M$.

**Exercise 1.3.** Let $M$ be a $k \times l$ matrix. There is an associated linear transformation $T_M: \mathbb{R}^l \rightarrow \mathbb{R}^k$. Minors $M'$ of $M$ which are $r \times r$ can be represented as a similar linear transformation; let $j: \mathbb{R}^r \hookrightarrow \mathbb{R}^l$ be the anti-projection transformation discussed in class, and $\pi: \mathbb{R}^k \rightarrow \mathbb{R}^r$ be the projection transformation discussed in class. Then $M' = [\pi \circ T_m \circ j]_{e_1, \ldots, e_n}$.

Now re-prove that $\text{rk}(M) = \max\{r \mid \det M' \neq 0 \text{ where } M' \text{ is an } r \times r \text{ minor of } M\}$ using the language of linear transformations discussed above.

**Exercise 1.4.** Show that for every square matrix $A$ there exists a single entry $a_{ij}$ which can be changed so that the rank of $A$ can be changed.

*Hint:* Do this in two parts. 1) If $M$ is a non-singular matrix show that it is possible to make it singular in this way. 2) If $M$ is singular, show that it is possible to increase its rank in this way.