2.1. Week 2 Problem Session 1.

Exercise 2.1. Let \( A \in M_n(\mathbb{R}) \). Let \( v \in \mathbb{R}^n \). We can define the linear map \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) where the \( i \)th component of the vector \( A(v) \) maps to:

\[
[A(v)]_i = [Av]_i = \sum_{j=1}^{n} a_{ij}v_j.
\]

Show that this gives a bijection between \( n \times n \) matrices and linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

Exercise 2.2. Prove the following formula for matrix multiplication by relating each matrix to a linear transformation: the \( ik \)th component of the matrix product \( AB \) is:

\[
(AB)_{ik} = \sum_{l=1}^{n} a_{il}b_{lk}.
\]

Exercise 2.3. Define the rank of a matrix in terms of the kernel of the associated linear map.

Exercise 2.4. Let \( A \) and \( B \) be \( n \times n \) matrices. We have shown the following relation:

\[
\dim(\text{column space of } A + B) \leq \dim(\text{span}(v_1, v_2, \ldots, v_k)) \leq k + l,
\]

where \( \{v_1, \ldots, v_k\} \) is a basis of the column space of \( A \) and \( \{w_1, \ldots, w_l\} \) is a basis of the column space of \( B \). Determine when the equalities are strict (that is, strictly less than).

Exercise 2.5. What is the characteristic polynomial of the \( n \times n \) matrix whose every entry is 1?

2.2. Week 2 Problem Session 2.

Exercise 2.6. Suppose \( \varphi : V \rightarrow W \) is a linear map, and some vector \( v \in V \). Let \( E = \{\vec{e}_1, \ldots, \vec{e}_k\} \) be the basis of \( V \), and \( F = \{\vec{f}_1, \ldots, \vec{f}_l\} \) a basis of \( W \). We will denote a vector \( v \) in coordinate form with respect to its basis by \([v]_E\). We denote the linear map \( \varphi \) by the matrix \([\varphi]_{E,F}\). Show that:

\[
[\varphi(v)]_F = [\varphi]_{E,F} \cdot [v]_E,
\]

where \([\varphi(v)]_F\) is a \( k \times 1 \) matrix, \([\varphi]_{E,F}\) is a \( k \times l \) matrix, and \([v]_E\) is a \( 1 \times 1 \) matrix.

Exercise 2.7. Let \( A, B \in F^{k \times l} \), and \( x \in F^l \). While it is not the case that if \( Ax = Bx \), then \( A = B \), prove that if \( \forall x \in F^l, Ax = Bx \), then \( A = B \).

Exercise 2.8. Use the previous two results to show that the composition of two linear maps corresponds to matrix multiplication. Be sure to carefully distinguish between the coordinate-representation of a vector and the vector itself. That is, we can say that a vector \( v \in V \) is associated with the coordinate \([x] = (x_1, \ldots, x_n) \in F^n\). However, we cannot say that \( v = [x] \); rather, we really mean that:

\[
v = \sum_{i=1}^{n} x_i \vec{e}_i.
\]

Exercise 2.9. In \( \mathbb{R}^2 \), with the standard basis \( \vec{e}_1, \vec{e}_2 \), we define a new basis \( \vec{f}_1 \) and \( \vec{f}_2 \), where \( \vec{f}_1 = \vec{e}_1 \), and \( \vec{f}_2 \) and \( \vec{f}_1 \) make an angle of degree \( \theta \). Let \( R \) be the transformation of the plane so that it rotates everything counterclockwise by \( \theta \). So, the basis vector \( \vec{f}_1 \) is mapped to where \( \vec{f}_2 \) was. Thus, the matrix will look like:

\[
[R]_{F,F} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Complete the matrix. Determine the same matrix in the standard basis, \([R]_{E,E}\). Compare the determinant and trace of both of these representations.