Final Exam: THURSDAY, March 17, 10:30-12:30.

Overview. The exam will have four questions (choose four out of five) and a nontrivial extra credit problem. Most of the final exam (at least three of the five questions) will cover material since the midterm. See the Midterm Review Sheet for earlier material.

In the last five weeks, a substantial theory was developed in order to obtain the proofs of our main theorems: Gödel’s two incompleteness theorems, Tarski’s theorem, Löb’s theorem and their corollaries, covered in Enderton §3.3-3.7. Know the statements and, within reason, the proofs of these results.

The purpose of exam questions on this material will be to test your understanding of these major results, e.g. what machinery and ingredients are necessary for the proofs, what hypotheses are required, the scope of their applicability, and basic consequences. When writing the exam, I will look for inspiration at: the homework, the review questions below, the main results proved in class, and exercises from Enderton. Note that the Paris-Kirby-Harrington methodology/results about incompleteness in Peano arithmetic will not be on the exam itself, though they may appear in challenge problems.

References: Class notes, Enderton chapter 3; Cohen is an enjoyable further reference.

1. Review Problems

These problems include some results found in the text. It’s recommended that you test yourself without looking at the book. $\mathcal{L}$ is the language of $\mathcal{N}$. Axiomatizable always means computably axiomatizable. A complete axiomatization is one whose set of consequences is complete. $T_E = Cn A_E$.

(1) Give a necessary and sufficient condition for a formula $\varphi$ to represent a relation $R$ in $T_E$, and prove this works.

(2) Prove that the set $\{\#\varphi : \varphi \in Th(\mathcal{N})\}$ is not definable in $\mathcal{N}$.

(3) Prove that there is no complete axiomatization of $Th(\mathcal{N})$.

(4) Given any $A \subset Th(\mathcal{N})$ such that $\#A$ is representable, give an example of a sentence $\sigma$ which is true in $\mathcal{N}$ but not a consequence of $A$.

(5) Prove that the set of Gödel numbers of valid formulas of $\mathcal{L}$ is not representable.

(6) Use the strong undecidability of $T_E$ to deduce that $Cn PA$ is not a complete theory.

(7) Let $A$ be a given set of axioms consistent with $A_E$. Assuming that the set $S$ of sequence numbers, the set $X$ of Gödel numbers of logical axioms, and the set $Y$ of Gödel numbers of elements of $A$ are each representable, prove that the set $C$ of Gödel numbers of consequences of $A$ is definable but is not representable.

(8) Show that if $T$ is axiomatizable and $A_E \subseteq T$, then for any sentence $\sigma$, $T \vdash \sigma$ implies $T \vdash Prb_T \sigma$. 
(9) Show that it is not true that for every sentence $\sigma$, $A_E \vdash (\sigma \rightarrow Prb_{A_E}\sigma)$.

(10) State Löb’s theorem and use it to derive Gödel’s second incompleteness theorem.

(11) Fixing some background language, say that the theory $T'$ is a completion of the theory $T$ if $T \subseteq T'$ and $T'$ is complete. Show that there is a theory $T$ which has both a decidable completion and an undecidable one.

(12) Show that every partial computable function has infinitely many indices.

(13) Show that a nonempty set is c.e. (i.e., it is the domain of a computable function) iff it is the range of a computable function, and that a c.e. set is computable iff both it and its complement are c.e.

(14) Prove that a set $C$ is c.e. iff it can be defined in $\mathcal{N}$ by an $L$-formula of the form $\exists y \psi_C(x,y)$, where $\psi_C(x,y)$ represents a computable relation.

(15) Suppose that $C$ can be defined in two different ways: (1) by a $L$-formula of the form $\exists y \psi(x,y)$ and (2) by a $L$-formula of the form $\forall y \varphi(x,y)$, where both $\psi(x,y)$ and $\varphi(x,y)$ represent computable relations. What can you conclude about $C$?

(16) Assume that all c.e. sets are diophantine. Prove that there is a diophantine set which is not computable.

(17) Let $M$ be any $L$-structure whose domain is $\mathbb{N}$. Prove that if $M \models T_E$ and $T = Th(M)$, then $\# T$ is not c.e.

(18) The sets $A, B$ are computably inseparable if there is no computable set $C$ such that $A \subseteq C$ and $C \cap B = \emptyset$. Show that there exist disjoint c.e. sets $A, B$ which are computably inseparable.

(19) Let $T$ be the theory of an equivalence relation with a class of size $n$ for each $n$, in the language with equality, a single binary edge relation and a constant, interpreted in the class of size 1. Does $T$ eliminate quantifiers?

(20) Give an example of an infinite indiscernible sequence in a model of $Th(\mathbb{N},0,S)$ and in a model of $Th(\mathbb{Q},<)$.

(21) Prove that for any infinite model $M$, there is $N \equiv M$ which contains an infinite indiscernible sequence.

(22) Let $M$ be the standard model of the theory $T$ from 19. Apply the previous question to get a quick proof that there exists $N' \equiv M$ in which there is an infinite equivalence class, and that there exists $N \equiv M$ in which there are infinitely many distinct infinite equivalence classes.