(1) *How sentences capture models.*

(a) Write down a single sentence of first order logic in the language \{<\} which expresses that \(<\) is a dense linear order without endpoints.

(b) Can this sentence have any finite models? (Justify your answer.)

(c) Finally, give two dense linear orders without endpoints of the same infinite size which are not isomorphic.

(2) *How formulas capture sets.* For each of the following relations, give a formula which defines it in \((\mathbb{N}; +, \times)\), where +, \times are binary function symbols interpreted as addition and multiplication.

(a) \{0\}

(b) \{1\}

(c) \{\langle m, n \rangle : n \text{ is the successor of } m \text{ in } \mathbb{N}\}

(d) \{\langle m, n \rangle : m < n \text{ in } \mathbb{N}\}

(3) *How formulas fail to capture sets.*

(a) Consider the model \((\mathbb{Z}; <)\) where < is interpreted as the usual order on the integers. Prove that the set of even numbers is not definable.

Bonus Consider the model \((\mathbb{Z}; <, 0)\) where in addition 0 is interpreted as the constant 0. Prove that the set of even numbers is not definable.

(4) Give a sentence having models of size 2^n for every positive integer n, but no finite models of odd size. (c.f. Enderton p. 101) Prove your answer works.

(5) *Deductions.* Show that any consistent set \(\Gamma\) of formulas can be extended to a consistent set \(\Delta\) having the property that for any formula \(\varphi\), either \(\varphi \in \Delta\) or \((\neg \varphi) \in \Delta\). [You may assume the language is countable. Do not use the compactness theorem for sentential logic, but you may quote results from Enderton §2.5.]

Challenge Problem. Say that an element \(a \in \text{dom}(M)\) is *definable* if \(\{a\}\) is a definable set. Give an example of a finite language \(\mathcal{L}\) and an infinite \(\mathcal{L}\)-model in which there is exactly one non definable element.