Lemma 11.1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous at \( p \in (a, b) \). Define functions \( m \) and \( M \) by:

\[
m(h) = \begin{cases} 
\inf\{ f(x) \mid p \leq x \leq p + h \} & \text{if } h \geq 0, \\
\inf\{ f(x) \mid p + h \leq x \leq p \} & \text{if } h < 0
\end{cases}
\]

\[
M(h) = \begin{cases} 
\sup\{ f(x) \mid p \leq x \leq p + h \} & \text{if } h \geq 0, \\
\sup\{ f(x) \mid p + h \leq x \leq p \} & \text{if } h < 0
\end{cases}
\]

Then \( \lim_{h \to 0} m(h) = f(p) \) and \( \lim_{h \to 0} M(h) = f(p) \).

Theorem 11.2 (The First Fundamental Theorem of Calculus). Suppose that \( f \) is integrable on \([a, b]\). Define \( F : [a, b] \rightarrow \mathbb{R} \) by

\[
F(x) = \int_a^x f.
\]

If \( f \) is continuous at \( p \in (a, b) \), then \( F \) is differentiable at \( p \) and

\[
F'(p) = f(p).
\]

Lemma 11.3. Suppose that \( f : [a, b] \rightarrow \mathbb{R} \) is integrable and that \( I \) is a number satisfying

\[
L(f, P) \leq I \leq U(f, P) \quad \text{for every partition } P \text{ of } [a, b].
\]

Then

\[
\int_a^b f = I.
\]

Theorem 11.4 (The Second Fundamental Theorem of Calculus). Let \( f \) be integrable on \([a, b]\). Suppose that there is a function \( F \) that is continuous on \([a, b]\) and differentiable on \((a, b)\) and such that \( f = F' \) on \((a, b)\). Then

\[
\int_a^b f = F(b) - F(a).
\]

Corollary 11.5 (Integration by Parts). Let \( f, g \) be functions defined on some open interval containing \([a, b]\) such that \( f' \) and \( g' \) exist and are continuous on \([a, b]\). Then

\[
\int_a^b fg' = [f(b)g(b) - f(a)g(a)] - \int_a^b f'g.
\]
Corollary 11.6 (Change of Variables). Let \( g \) be a function defined on some open interval containing \([a, b]\) such that \( g' \) is continuous on \([a, b]\). Suppose that \( g([a, b]) \subset [c, d] \) and \( f : [c, d] \longrightarrow \mathbb{R} \) is continuous. Define \( F : [c, d] \longrightarrow \mathbb{R} \) by \( F(x) = \int_c^x f \). Then

\[
\int_a^b f(g(x)) \cdot g'(x) \, dx = F(g(b)) - F(g(a)).
\]

Now, we prove another very important theorem that tells us about inverse functions and their derivatives. To get there we will need a few lemmas.

Exercise 11.7. Show that if \( f \) is strictly increasing or strictly decreasing on an interval, then \( f \) is injective.

Lemma 11.8. If \( f : (a, b) \rightarrow \mathbb{R} \) is continuous and injective, then \( f \) is either strictly increasing or strictly decreasing on \((a, b)\).

Hint: Assume \( f : (a, b) \rightarrow \mathbb{R} \) is continuous and injective. Fix two points \( x_1, x_2 \in (a, b) \) with \( x_1 < x_2 \). Let \( y, z \in (a, b) \) be any two points with \( y < z \). Define \( h : [0, 1] \longrightarrow \mathbb{R} \) by \( h(t) = f((1-t)x_2 + tz) - f((1-t)x_1 + ty) \). Consider \( h(0) \) and \( h(1) \) and show that they must have the same sign.

In Lemma 1.28 we saw that if \( f : A \longrightarrow B \) is bijective then there is a bijection \( g : B \longrightarrow A \), called the inverse function, such that \( (g \circ f)(a) = a, \forall a \in A \), and \( (f \circ g)(b) = b, \forall b \in B \).

Theorem 11.9. If \( f : (a, b) \rightarrow \mathbb{R} \) is continuous and injective, then the inverse function \( g : f(a, b) \rightarrow (a, b) \) is continuous.

We denote the inverse function \( g \) by \( f^{-1} \).

Theorem 11.10. Suppose that \( f : (a, b) \rightarrow \mathbb{R} \) is differentiable and that the derivative \( f' : (a, b) \rightarrow \mathbb{R} \) is continuous. Also suppose that there is a point \( p \in (a, b) \) such that \( f'(p) \neq 0 \). Then there exists a region \( R \subset (a, b) \) such that \( p \in R \) and \( f \) with domain restricted to \( R \) is injective. Furthermore, \( f^{-1} : f(R) \rightarrow R \) is differentiable at the point \( f(p) \) and

\[
(f^{-1})'(f(p)) = \frac{1}{f'(p)}.
\]

Hint: Remark 9.35 and Lemma 9.23 should be useful.

Exercise 11.11. Consider the function \( f(x) = x^n \) for a fixed \( n \in \mathbb{N} \). If \( n \) is even, then \( f \) is strictly increasing on the set of non-negative real numbers. If \( n \) is odd, then \( f \) is strictly increasing on all of \( \mathbb{R} \). For a given \( n \), let \( A \) be the aforementioned set on which \( f \) is strictly increasing. Define the inverse function \( f^{-1} : f(A) \rightarrow A \) by \( f^{-1}(x) = \sqrt[n]{x} \), which we sometimes also denote \( f^{-1}(x) = x^{1/n} \). Use Theorem 11.10 to find the points \( y \in f(A) \) at which \( f^{-1} \) is differentiable, and determine \( (f^{-1})'(y) \) at these points.