We will now consider a notion of continuity that is stronger than ordinary continuity.

**Definition 10.1.** Let \( f: A \rightarrow \mathbb{R} \) be a function. We say that \( f \) is uniformly continuous if for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x, y \in A \)

\[
\text{if } |x - y| < \delta, \quad \text{then } |f(x) - f(y)| < \epsilon.
\]

**Theorem 10.2.** If \( f \) is uniformly continuous, then \( f \) is continuous.

**Exercise 10.3.** Determine with proof whether the following functions \( f \) are uniformly continuous on the given intervals \( A \):

1. \( f(x) = x^2 \) on \( A = \mathbb{R} \)
2. \( f(x) = x^2 \) on \( A = (-2, 2) \)
3. \( f(x) = \frac{1}{x} \) on \( A = (0, +\infty) \)
4. \( f(x) = \frac{1}{x} \) on \( A = [1, +\infty) \)
5. \( f(x) = \sqrt{x} \) on \( A = [0, +\infty) \)
6. \( f(x) = \sqrt{x} \) on \( A = [1, +\infty) \)

**Exercise 10.4.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^n \), for \( n \in \mathbb{N} \). Show that \( f \) is uniformly continuous if, and only if, \( n = 1 \).

**Challenge:** Let \( p : \mathbb{R} \rightarrow \mathbb{R} \) be a polynomial with real coefficients. Show that \( p \) is uniformly continuous on \( \mathbb{R} \) if and only if \( \deg(p) \leq 1 \).

**Exercise 10.5.** Let \( f \) and \( g \) be uniformly continuous on \( A \subset \mathbb{R} \). Show that:

1. The function \( f + g \) is uniformly continuous on \( A \).
2. For any constant \( c \in \mathbb{R} \), the function \( c \cdot f \) is uniformly continuous on \( A \).

We will now prove that continuous functions with compact domain are automatically uniformly continuous. To this end, first consider:

**Lemma 10.6.** Let \( f: A \rightarrow \mathbb{R} \) be continuous. Fix \( \epsilon > 0 \). By the definition of continuity, for each \( p \in A \) there exists \( \delta(p) > 0 \) such that for all \( x \in A \)

\[
\text{if } |x - p| < \delta(p), \quad \text{then } |f(x) - f(p)| < \frac{\epsilon}{2}.
\]

For each \( p \in A \), define \( U(p) = \{ x \in \mathbb{R} \mid |x - p| < \frac{1}{2} \delta(p) \} \). Then the collection \( \{ U(p) \mid p \in A \} \) is an open cover of \( A \).
**Theorem 10.7.** Suppose that $X \subset \mathbb{R}$ is compact and $f : X \rightarrow \mathbb{R}$ is continuous. Then $f$ is uniformly continuous.

**Corollary 10.8.** Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then $f$ is uniformly continuous.

**Definition 10.9.** We say that a function $f : A \rightarrow \mathbb{R}$ is bounded if $f(A)$ is a bounded subset of $\mathbb{R}$.

**Exercise 10.10.** Show that if $X \subset \mathbb{R}$ is compact and $f : X \rightarrow \mathbb{R}$ is continuous, then $f$ is bounded.

**Exercise 10.11.** Show that if $f$ and $g$ are bounded on $A$ and uniformly continuous on $A$, then $fg$ is uniformly continuous on $A$.

We are now ready to turn to integration.

**Definition 10.12.** A partition of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes $a$ and $b$: $$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$ If $P$ and $Q$ are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to $Q$ as a refinement of $P$.

We usually write partitions as ordered lists $P = \{t_0, t_1, \ldots, t_n\}$ with $t_{i-1} < t_i$ for each $i = 1, \ldots, n$.

**Definition 10.13.** Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded and that $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$. Define:

- $m_i = \inf \{f(x) \mid t_{i-1} \leq x \leq t_i\}$
- $M_i = \sup \{f(x) \mid t_{i-1} \leq x \leq t_i\}$

The lower sum of $f$ for the partition $P$ is the number:

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of $f$ for the partition $P$ is the number:

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Notice that it is always the case that $L(f, P) \leq U(f, P)$.

**Lemma 10.14.** Suppose that $P$ and $Q$ are partitions of $[a, b]$ and that $Q$ is a refinement of $P$. Then:

$$L(f, P) \leq L(f, Q) \quad \text{and} \quad U(f, P) \geq U(f, Q).$$
Theorem 10.15. Let $P_1$ and $P_2$ be partitions of $[a,b]$ and suppose that $f: [a,b] \to \mathbb{R}$ is bounded. Then:

$$L(f, P_1) \leq U(f, P_2).$$

Definition 10.16. Let $f: [a,b] \to \mathbb{R}$ be bounded. We define:

$$L(f) = \sup \{ L(f, P) \mid P \text{ is a partition of } [a,b] \}$$

$$U(f) = \inf \{ U(f, P) \mid P \text{ is a partition of } [a,b] \}$$

to be, respectively, the lower integral and upper integral of $f$ from $a$ to $b$.

Exercise 10.17. Why do $L(f)$ and $U(f)$ exist? Find a function $f$ for which $L(f) = U(f)$. Find a function $f$ for which $L(f) \neq U(f)$. Is there a relationship between $L(f)$ and $U(f)$ in general?

Definition 10.18. Let $f: [a,b] \to \mathbb{R}$ be bounded. We say that $f$ is integrable on $[a,b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the integral of $f$ from $a$ to $b$ and we write it as:

$$\int_a^b f.$$

When we want to display the variable of integration, we write the integral as follows, including the symbol $dx$ to indicate that variable of integration:

$$\int_a^b f(x) \, dx.$$

For example, if $f(x) = x^2$, we would write $\int_a^b x^2 \, dx$ but not $\int_a^b x^2$.

Theorem 10.19. Let $f: [a,b] \to \mathbb{R}$ be bounded. Then $f$ is integrable if and only if for every $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Theorem 10.20. If $f: [a,b] \to \mathbb{R}$ is continuous, then $f$ is integrable.

(Hint: Use Theorem 10.19 and uniform continuity.)

Exercise 10.21. Fix $c \in \mathbb{R}$ and let $f: [a,b] \to \mathbb{R}$ be defined by $f(x) = c$, for each $x \in [a,b]$. Show that $f$ is integrable on $[a,b]$ and that $\int_a^b f = c(b-a)$.

Lemma 10.22. Let $f: [a,b] \to \mathbb{R}$ be bounded. Given $I \in \mathbb{R}$, $I = \int_a^b f$ if, and only if, for all $\epsilon > 0$, there is some partition $P$ such that

$$U(f, P) - I < \epsilon \quad \text{and} \quad I - L(f, P) < \epsilon.$$
Exercise 10.23. Define $f : [0, b] \to \mathbb{R}$ by the formula $f(x) = x$. Show that $f$ is integrable on $[0, b]$ and that $\int_0^b f = \frac{b^2}{2}$.

Exercise 10.24. Show that the converse of Theorem 10.20 is false in general.

Theorem 10.25. Let $a < b < c$. A function $f : [a, c] \to \mathbb{R}$ is integrable on $[a, c]$ if and only if $f$ is integrable on $[a, b]$ and $[b, c]$. When $f$ is integrable on $[a, c]$, we have

$$\int_a^c f = \int_a^b f + \int_b^c f.$$ 

If $b < a$, we define

$$\int_a^b f = -\int_b^a f,$$

whenever the latter integral exists. With this notational convention, it follows that the equation

$$\int_a^c f = \int_a^b f + \int_b^c f$$

always holds, regardless of the ordering of $a$, $b$ and $c$, whenever $f$ is integrable on the largest of the three intervals.

Theorem 10.26. Suppose that $f$ and $g$ are integrable functions on $[a, b]$ and that $c \in \mathbb{R}$ is a constant. Then $f + g$ and $cf$ are integrable on $[a, b]$ and

(i) $\int_a^b (f + g) = \int_a^b f + \int_a^b g$, 

(ii) $\int_a^b c \cdot f = c \int_a^b f$. 

Theorem 10.27. Suppose that $f$ is integrable on $[a, b]$. Then there exist numbers $m$ and $M$ such that:

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Theorem 10.28. Suppose that $f$ is integrable on $[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f.$$

Then $F$ is continuous.