In this sheet we give the continuum $C$ a topology. Roughly speaking, this is a way to describe how the points of $C$ are ‘glued together’.

**Definition 3.1.** A subset of the continuum is *closed* if it contains all of its limit points.

**Theorem 3.2.** The sets $\emptyset$ and $C$ are closed.

**Theorem 3.3.** A subset of $C$ containing a finite number of points is closed.

**Definition 3.4.** Let $X$ be a subset of $C$. The *closure* of $X$ is the subset $\overline{X}$ of $C$ defined by:

$$
\overline{X} = X \cup \{x \in C \mid x \text{ is a limit point of } X\}.
$$

**Theorem 3.5.** $X \subset C$ is closed if and only if $X = \overline{X}$.

**Theorem 3.6.** The closure of $X \subset C$ satisfies $\overline{X} = \overline{\overline{X}}$.

**Corollary 3.7.** Given any subset $X \subset C$, the closure $\overline{X}$ is closed.

**Definition 3.8.** A subset $G$ of the continuum is *open* if its complement $C \setminus G$ is closed.

**Theorem 3.9.** The sets $\emptyset$ and $C$ are open.

The following is a very useful criterion to determine whether a set of points is open.

**Theorem 3.10.** Let $G \subset C$. Then $G$ is open if and only if for all $x \in G$, there exists a region $R$ such that $x \in R \subset G$.

**Corollary 3.11.** Every region $R$ is open. Every complement of a region, $C \setminus R$, is closed.

**Corollary 3.12.** Let $G \subset C$. Then $G$ is open if and only if for all $x \in G$, there exists a subset $V \subset G$ such that $x \in V$ and $V$ is open.

**Corollary 3.13.** Let $a \in C$. Then the sets $\{x \mid x < a\}$ and $\{x \mid a < x\}$ are open.

**Theorem 3.14.** Let $G$ be a nonempty open set. Then $G$ is the union of a collection of regions.

**Exercise 3.15.** Do there exist subsets $X \subset C$ that are neither open nor closed?

**Theorem 3.16.** Let $\{X_\lambda\}$ be an arbitrary collection of closed subsets of the continuum. Then the intersection $\bigcap_\lambda X_\lambda$ is closed.

**Theorem 3.17.** Let $G_1, \ldots, G_n$ be a finite collection of open subsets of the continuum. Then the intersection $G_1 \cap \cdots \cap G_n$ is open.
Exercise 3.18. Is it necessarily the case that the intersection of an infinite number of open sets is open? Is it possible to construct an infinite collection of open sets whose intersection is not open? Equivalently, is it possible to construct an infinite collection of closed sets whose union is not closed?

Corollary 3.19. Let \( \{G_\lambda\} \) be an arbitrary collection of open subsets of the continuum. Then the union \( \bigcup_\lambda G_\lambda \) is open. Let \( X_1, \ldots, X_n \) be a finite collection of closed subsets of the continuum. Then the union \( X_1 \cup \cdots \cup X_n \) is closed.

Theorem 3.14 says that every nonempty open set is the union of a collection of regions. This necessary condition for open sets is also sufficient:

Corollary 3.20. Let \( G \subset C \) be nonempty. Then \( G \) is open if and only if \( G \) is the union of a collection of regions.

Corollary 3.21. If \( ab \) is a region in \( C \), then \( \text{ext} \ ab \) is open.

Theorem 3.9 and Corollary 3.19 say that the collection \( \mathcal{T} \) of open subsets of the continuum is a topology on \( C \), in the following sense:

Definition 3.22. Let \( X \) be any set. A topology on \( X \) is a collection \( \mathcal{T} \) of subsets of \( X \) that satisfy the following properties:

1. \( X \) and \( \emptyset \) are elements of \( \mathcal{T} \).
2. The union of an arbitrary collection of sets in \( \mathcal{T} \) is also in \( \mathcal{T} \).
3. The intersection of a finite number of sets in \( \mathcal{T} \) is also in \( \mathcal{T} \).

The elements of \( \mathcal{T} \) are called the open sets of \( X \). The set \( X \) with the structure of the topology \( \mathcal{T} \) is called a topological space\(^1\).

Definition 3.23. A topological space \( X \) is discrete if every subset of \( X \) is open.

Exercise 3.24. Find a realization of the continuum that is discrete. Must every realization be discrete?

The following definition introduces the concept of “connectedness” of an abstract topological space. In order to define this properly, we need to understand what “closure” means in an abstract topological space. In turn, in order to understand this, we need to understand what “limit points” are in an abstract topological space. The following definition gives us this information. You should verify that these definitions are consistent with Definitions 2.11 and 3.4; they simply extend them to the case of abstract topological spaces. In other words, Definitions 2.11 and 3.4 can now be thought of as special cases of Definition 3.25.

\(^1\)The word topology comes from the Greek word *topos* (τόπος), which means “place”.
Definition 3.25. Let $X$ be a topological space and let $A \subset X$. We say that $a \in X$ is a \emph{limit point} of $A$ if

for every open subset $U$ of $X$ such that $a \in U$, we have $U \cap (A \setminus \{a\}) \neq \emptyset$.

As before, we define

$$LP(A) := \{a \in X \mid a \text{ is a limit point of } A\} \quad \text{and} \quad \overline{A} := A \cup LP(A).$$

Definition 3.26. Let $X$ be a topological space and let $Y \subset X$ (Note: It is permissible for $Y$ to equal $X$ here). We say that two subsets $A$ and $B$ of $X$ form a \emph{separation} of $Y$ if

1. $A \neq \emptyset \neq B$;
2. $A \cup B = Y$; and
3. $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

We say that $Y$ is disconnected if there exists a separation of $Y$. We say that $Y$ is connected if $Y$ is not disconnected.

Exercise 3.27. Let $a \in C$. Prove that $C \setminus \{a\}$ is disconnected.

Exercise 3.28. Prove by direct example that both $\mathbb{Z}$ and $\mathbb{Q}$ are disconnected realizations of the continuum. Explain (informally, but in logical, coherent sentences) why neither of your proofs could be used to show that every realization of the continuum is disconnected.