Abstract. These are notes on the construction of the rational numbers from the integers. We go through equivalence relations, the definition of rationals, addition and multiplication of rationals, the ordering on the rational numbers, and the countability of the rationals.

Now that we have defined the integers, the next step is to construct the rational numbers. Intuitively, the rational numbers are just quotients of integers by nonzero integers. What we would like to do is define a fraction by its numerator and denominator \( \frac{1}{2}, \frac{3}{4}, \text{etc.} \), but with the caveat that expressions such as \( \frac{1}{4} \) and \( \frac{2}{8} \) are the same. To do this, we need the notion of equivalence relations.

1. Equivalence relations

Definition 1.1. Let \( X \) be a set. An equivalence relation on \( X \) is a subset \( \sim \) of \( X \times X \) (with \( (x, y) \in \sim \) written as \( x \sim y \)) satisfying the following three properties:

a. (Reflexivity) For all \( x \in X \), we have \( x \sim x \).

b. (Symmetry) For all \( x, y \in X \), if \( x \sim y \), then \( y \sim x \).

c. (Transitivity) For all \( x, y, z \in X \), if \( x \sim y \) and \( y \sim z \), then \( x \sim z \).

Remark. We will use the notation \( a \sim b \sim c \) to mean \( a \sim b \) and \( b \sim c \). We will use similar notation for longer chains of equivalences.

Example 1.2. The first example of an equivalence relation is equality. It is true that \( a = a \), that \( a = b \) implies \( b = a \), and that \( a = b \) and \( b = c \) implies \( a = c \). One can think of an equivalence relation as a generalized equality relation, where intuitively, \( x \sim y \) means that \( x \) and \( y \) are identified.

Example 1.3. For a less trivial example, let \( X = \mathbb{Z} \), and set \( x \sim y \) if \( x - y \) is even. One can easily check the three properties for this relation.

Definition 1.4. Let \( \sim \) be an equivalence relation on \( X \). For any given \( x \in X \), let \( [x] \) be the subset of \( X \) given by

\[ [x] = \{ y \in X | x \sim y \}. \]

A subset of \( X \) is called an equivalence class if it is equal to \([x]\) for some \( x \in X \). We call \([x]\) the equivalence class of \( x \).

Example 1.5. If our equivalence relation were equality (Example 1.2), then \([x] = \{x\}\). In the example \( X = \mathbb{Z} \) with \( x \sim y \) if \( x - y \) is even (Example 1.3), we can see that \([1] = [3] = \{x \in \mathbb{Z} | x \text{ is odd}\} \) and \([0] = [2] = \{x \in \mathbb{Z} | x \text{ is even}\} \).

Remark. The notation of using \([x]\) for the equivalence class of \( x \) is standard but conflicts with our notation of using \([n]\) for \( \{1, \cdots, n\} \). Please use context to determine which of the notations is in use.

Definition 1.6. Given an equivalence relation \( \sim \) on a set \( X \), we denote the set of all equivalence classes by \( X/\sim \). There is a natural function \( X \to X/\sim \) given by sending an element \( x \) to its equivalence class \([x]\).
Example 1.7. In Example 1.3, the set \( \mathbb{Z}/\sim \) would consist of \( \{[1], [2]\} \).

Lemma 1.8. Given an equivalence relation \( \sim \) on a set \( X \), the natural function \( X \to X/\sim \) is a surjection.

Proof. By the definition of equivalence class, any given equivalence class is equal to \([x]\) for some \( x \in X \).

Lemma 1.9. Let \( \sim \) be an equivalence relation on a set \( X \). Every element \( x \in X \) is contained in a unique equivalence class.

Proof. Each \( x \in X \) is contained in \([x]\), so every element is contained in some equivalence class.

To show uniqueness, suppose \( x \in [y] \) and \( x \in [z] \), so that \( y \sim x \) and \( z \sim x \). We claim that \([y] \subset [z]\). Indeed, if \( w \in [y] \), then \( y \sim w \). Then by symmetry and transitivity, we have \( z \sim w \). Thus \([y] \subset [z]\). By symmetry, we also have \([z] \subset [y]\). Therefore \([y] = [z]\), as desired.

Corollary 1.10. We have \([x] = [y]\) if and only if \( x \sim y \).

Proof. If \([x] = [y]\), then \( y \in [y] = [x] \), so \( x \sim y \).

If \( x \sim y \), then both \([x]\) and \([y]\) contain \( y \). Because \( y \) is contained in exactly one equivalence class, we conclude that \([x] = [y]\).

2. Definition of the Rationals

Now that we have equivalence relations, we wish to define a rational number to be a pair \((a, b)\) where \(a\) is the numerator and \(b\) is the denominator up to an equivalence relation. To see what our equivalence relation should be, note that \( \frac{a}{b} \) should equal \( \frac{c}{d} \) precisely when \( ad - bc = 0 \).

Lemma 2.1. Consider the set \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \). Then the relation \( \sim \) defined by \((a, b) \sim (c, d)\) if \( ad - bc = 0 \) is an equivalence relation.

Proof. For reflexivity, note that \((a, b) \sim (a, b)\) is equivalent to \( ab - ba = 0 \), which is true.

For symmetry, if \((a, b) \sim (c, d)\), then \( ad - bc = 0 \). But this means that \( da - cb = 0 \), so \((c, d) \sim (a, b)\).

Finally, for transitivity, if \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\), then \( ad - bc = 0 \) and \( ef - de = 0 \). Multiplying the first equation by \( f \) and the second equation by \( b \), we see that \( adf - bcf = 0 \) and \( bcf - bde = 0 \). Adding the two equations, we see that \( 0 = adf - bde = (af - be) \). Since \( d \neq 0 \), we conclude that \( af - be = 0 \), so \((a, b) \sim (e, f)\).

Definition 2.2. We define the rational numbers to be the set of equivalence classes \((\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})))/\sim \), where \( \sim \) is the equivalence relation given in the preceding lemma. We will use the notation \( \frac{a}{b} \) or \( a/b \) to denote the equivalence class \([a, b]\). The set of rationals is also denoted by \( \mathbb{Q} \).

Lemma 2.3. There is an injection \( \mathbb{Z} \to \mathbb{Q} \) given by sending \( n \) to \( n/1 \).

Proof. If \( n/1 = m/1 \), then \( 0 = n \cdot 1 - 1 \cdot m = n - m \), so \( n = m \).

In light of the injection, we often think of \( \mathbb{Z} \) as a subset of \( \mathbb{Q} \). We will then use the symbol \( n \) to denote \( n/1 \) when it is viewed as an element of \( \mathbb{Q} \). For example, we shall use 1 to denote 1/1.
3. Arithmetic of Rational Numbers

We now define addition and multiplication on the rational numbers. For addition
we should have \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \), while for multiplication, we should have \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \).

To make sure that these operations are well defined, we need to check that the
equivalence classes \( \frac{a}{b} \) and \( \frac{c}{d} \) do not depend on our choices of representatives
\((a, b)\) and \((c, d)\) of the equivalence classes \( \frac{a}{b} \) and \( \frac{c}{d} \), respectively.

**Lemma 3.1.** The operations + and \times given by \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \) and \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \) are well defined on the rationals.

**Proof.** We prove here that \( \times \) is well-defined; the proof of + is left as an exercise.

First, note that if \( b, d \neq 0 \), then \( bd \neq 0 \), so \( \frac{a}{b} \) is an element of \( \mathbb{Q} \).

Now suppose \( \frac{a}{b} = \frac{a'}{b'} \) and \( \frac{c}{d} = \frac{c'}{d'} \). We wish to show that \( \frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'} \).

We need to show that \( a_1c_1b_2d_2 = a_2c_2b_1d_1 \). Using the fact that \( a_1b_2 = b_2a_1 \) and \( c_1d_2 = c_2d_1 \), we see that

\[
\begin{align*}
a_1c_1b_2d_2 &= a_2c_2b_1d_1, \\
&= a_2c_2b_1d_1,
\end{align*}
\]
as desired.

Thus the operation of multiplication is well defined on the rationals. \( \square \)

**Theorem 3.2.** Addition and multiplication on the rationals satisfy the following properties
a. (Associativity of addition) For all \( x, y, z \in \mathbb{Q} \), we have \( (x + y) + z = x + (y + z) \).
 b. (Commutativity of addition) For all \( x, y \in \mathbb{Q} \), we have \( x + y = y + x \).
 c. (Additive identity) There exists an element \( 0' \in \mathbb{Q} \) such that for all \( x \in \mathbb{Q} \), we have \( 0' + x = x \).
 d. (Additive inverses) For every \( x \in \mathbb{Q} \), there exists an element \( -x \in \mathbb{Q} \) such that \( x + (\neg x) = 0' \).
 e. (Associativity of multiplication) For all \( x, y, z \in \mathbb{Q} \), we have \( (xy)z = x(yz) \).
 f. (Commutativity of multiplication) For all \( x, y \in \mathbb{Q} \), we have \( xy = yx \).
 g. (Multiplicative identity) There exists an element \( 1' \in \mathbb{Q} \) such that for all \( x \in \mathbb{Q} \),
we have \( 1' \cdot x = x \).
 h. (Multiplicative inverses) For all \( x \neq 0' \) in \( \mathbb{Q} \), there exists an \( x^{-1} \in \mathbb{Q} \) such that \( x \cdot x^{-1} = 1' \).
 i. (Distributivity of multiplication over addition) For all \( x, y, z \in \mathbb{Q} \), we have \( x(y + z) = xy + xz \).

**Proof.** (Left as an exercise.) \( \square \)

**Lemma 3.3.** The addition and multiplication of rationals extends addition and multiplication on the integers. More precisely, if \( i : \mathbb{Z} \rightarrow \mathbb{Q} \) is the injection \( n \mapsto n/1 \) of Lemma 2.3, then \( i(n + m) = i(n) + i(m) \) and \( i(nm) = i(m) \cdot i(n) \).

**Proof.** We prove the additive property here; the proof of the multiplicative property
is left as an exercise.

We see that

\[
i(n + m) = \frac{n + m}{1} = \frac{n \cdot 1 + 1 \cdot m}{1 \cdot 1} = \frac{n + m}{1} = i(n) + i(m),
\]
as desired. \( \square \)
Since addition and multiplication inside the integers and inside the rationals agree, there is no ambiguity when we write $n + m$ or $nm$.

4. Ordering the Rationals

**Lemma 4.1.** For every rational number $x$, there exists a positive integer $n$ such that $n \cdot x$ is an integer. That is, $n \cdot x = m = m/1$ for some integer $m$.

*Proof.* Let $x = a/b$. Take $n = |b|$. Then $|b| \cdot \frac{a}{b} = a \cdot \frac{|b|}{1}$. Since $b \neq 0$, it is either positive or negative. One can easily verify that $|b|/b$ is $1$ if $b > 0$ and $-1$ if $b < 0$. Thus $|b| \cdot \frac{a}{b} = a \cdot \frac{|b|}{b}$ is a product of integers and therefore an integer. □

**Corollary 4.2.** For a finite set of rationals $x_1, \cdots, x_m$, there exists a positive integer $n$ such that $n \cdot x_i$ is an integer for $i = 1, \ldots, m$.

*Proof.* By the lemma, we can find positive integers $n_i$ such that $n_i \cdot x_i$ is an integer. Take $n$ to be the product of the $n_i$, which is a positive integer since all the $n_i$ are positive integers. Then $n \cdot x_i = (n_i \cdot x_i) \cdot \prod_{j \neq i} n_j$, which is a product of integers, and therefore an integer. □

**Definition 4.3.** If $x, y \in \mathbb{Q}$, we say $x \prec y$ if there exists a positive integer $n$ such that $nx$ and $ny$ are integers and $nx < ny$.

*Remark.* We will only use the notation $\prec$ temporarily. After we show that $\prec$ is an order on $\mathbb{Q}$ which restricts to the usual order on $\mathbb{Z}$, we shall use $<$ instead.

**Lemma 4.4.** Suppose $x, y \in \mathbb{Q}$. Then $x \prec y$ if and only if, for every positive integer $m$ such that $mx$ and $my$ are integers, we have $mx < my$.

*Proof.* For the “only if” statement, the proof is by contradiction. Suppose there is a positive integer $m$ such that $mx$ and $my$ are integers and $mx \geq my$. Since $x \prec y$, we can find a positive integer $n$ such that $nx$ and $ny$ are integers and $nx < ny$. Because $n$ is positive, we can multiply the inequality $nx \geq ny$ by $n$ to see that $mnx \geq mny$. Similarly, since $m$ is positive, we can multiply the inequality $nx < ny$ by $m$ to see that $mnx < mny$. But then $mnx$ and $mny$ are integers such that $mnx \geq mny$ and $mnx < mny$, a contradiction.

For the “if” statement, we can find a positive integer $n$ such that $nx$ and $ny$ are integers by the preceding corollary (Corollary 4.2). Then $nx < ny$ by assumption, so $x \prec y$. □

**Theorem 4.5.** The relation $\prec$ is an ordering on $\mathbb{Q}$.

*Proof.* (Left as an exercise.) □

We have shown that $\prec$ is actually an ordering on $\mathbb{Q}$. Note that if $n, m$ are integers and $n < m$, then $1 \cdot n \leq 1 \cdot m$, so $n \prec m$. Thus the ordering $\prec$ extends the ordering $<$ on the integers. From now on, we drop the temporary notation $\prec$, and say $x < y$ to mean $x \prec y$ for two elements $x, y$ in $\mathbb{Q}$ instead.

**Theorem 4.6.** The following two properties hold:

a. If $x, y, c \in \mathbb{Q}$ and $x > y$, then $x + c > y + c$.

b. If $x > y$ and $c > 0$, then $xc > yc$.

*Proof.* (Left as an exercise.) □
5. COUNTABILITY OF THE RATIONALS

Given a set, one may ask “how many elements does it have?”. For finite sets, we have answered this question already: a set \( A \) has \( n \) elements if \( A \) and \([n]\) are in bijective correspondence. We now ask the same question for infinite sets.

**Definition 5.1.** We say two sets \( A \) and \( B \) have the same cardinality if \( A \) and \( B \) are in bijective correspondence. We denote this by \( |A| = |B| \). We use \( \aleph_0 \) (pronounced “aleph-not”) to denote the cardinality of \( \mathbb{N} \), that is, we write \( |\mathbb{N}| = \aleph_0 \) and \( |A| = \aleph_0 \) if \( A \) and \( \mathbb{N} \) are in bijective correspondence. We say a set is countable if it is finite or in bijective correspondence with \( \mathbb{N} \).

**Definition 5.2.** We define the relation \( \sim \) on the collection of all sets by saying \( A \sim B \) if there exists a bijection \( f : A \to B \).

**Proposition 5.3.** The relation \( \sim \) is an equivalence relation.

*Proof. (Left as an exercise.)*

Intuitively, if there exists a bijection between two sets, then they must “have the same number of elements”. While this makes some intuitive sense, there is a real danger when trying to transfer intuition from the finite world to the infinite world.

**Lemma 5.4.** \( \mathbb{Z} \sim \mathbb{N} \).

*Proof. (Left as an exercise.)*

From the example, we see that it is possible for \( A \) to be a proper subset of \( B \) and still have the same cardinality as \( B \). One thing people think is that since \( \mathbb{N} \subset \mathbb{Z} \), the set \( \mathbb{Z} \) must have “more elements” than \( \mathbb{N} \), and get confused why \( |\mathbb{Z}| = |\mathbb{N}| \). The problem comes when you try and formalize what “more elements” means. There is a well defined way in which \( \mathbb{Z} \) has “greater than” \( \mathbb{N} \), namely that of the subset relation. When we say “greater than” we are thinking of some concept of ordering and \( \subset \) is a well defined (partial) order on subsets of a set. However, this notion is not the notion we are looking for when we are counting. For example \{apple\} and \{orange\} both have one element, but neither is a subset of the other. When we count, we are indifferent to what the elements of the set are, we just care whether there are the “same number of them”. So when we are counting, we are indifferent to what the elements of \( \mathbb{N} \) and \( \mathbb{Z} \) are, we just care that there exists a bijection between the two sets.

In the following, we will construct some functions into and out of \( \mathbb{N} \). To do this we need the well-ordering principle: every nonempty subset of \( \mathbb{N} \) has a least element. If we are proving the well-ordering principle, we might as well prove the principle of strong induction on the way.

**Theorem 5.5** (Principle of Strong Induction). Suppose \( P(n) \) is a statement for \( n \in \mathbb{N} \). If

a. \( P(1) \) is true,

b. \( P(1) \land \cdots \land P(n) \) implies \( P(n+1) \),

then \( P(n) \) is true for all \( n \in \mathbb{N} \).

*Proof. Let \( Q(n) \) be the statement \( P(1) \land \cdots \land P(n) \). We prove \( Q(n) \) for all \( n \) by induction on \( n \).

The statement \( Q(1) \) is true because \( P(1) \) is true.
Suppose \( Q(n) \) is true. Then \( P(1), \ldots, P(n) \) are all true, so \( P(n+1) \) is true. But this means that \( Q(n+1) \) is true. By the principle of induction, we conclude that \( Q(n) \) is true for all \( n \in \mathbb{N} \).

Since \( Q(n) \) implies \( P(n) \), this means that \( P(n) \) is true for all \( n \in \mathbb{N} \).

\[ \square \]

**Theorem 5.6** (Well-ordering Principle). Every nonempty subset of \( \mathbb{N} \) contains a least element.

Proof. (Left as an exercise.) (Hint: Use strong induction.) \[ \square \]

**Theorem 5.7.** Every subset of \( \mathbb{N} \) is countable.

Proof. A subset of \( \mathbb{N} \) is either finite or infinite (that is, not finite). If it is finite, then we are done.

Suppose \( A \subset \mathbb{N} \) is infinite. We recursively define a bijection \( f : \mathbb{N} \to A \). Let \( f(1) \) be the least element of \( A \). By the well-ordering principle, this element exists because \( A \) is infinite, therefore nonempty. Now let \( f(n+1) \) be the least element of \( A \setminus \{f(1), \ldots, f(n)\} \). By well ordering, this exists because if \( A \setminus \{f(1), \ldots, f(n)\} \) were empty, then \( A \subset \{f(1), \ldots, f(n)\} \) must be finite. We have thus recursively defined a function \( f : \mathbb{N} \to A \).

We wish to show \( f \) is a bijection. To show it is injective, if \( n < m \), then \( f(m) \in A \setminus \{f(1), \ldots, f(m-1)\} \) cannot, by definition, be equal to \( f(n) \). Thus the function is injective. To show the function is surjective, suppose \( a \in A \) is not equal to \( f(n) \) for any \( n \in \mathbb{N} \). Then by definition of \( f(n) \), we have \( a > f(n) \) for every \( n \in \mathbb{N} \). We claim that \( f(n) \geq n \) for all \( n \in \mathbb{N} \). Indeed, if not, the restriction of \( f \) to \([n]\) can be viewed as an injective function \([n] \to [f(n)]\) where \( n > f(n) \), contradicting the pigeonhole principle. But this means that \( a > f(a+1) \geq a + 1 \), which is clearly absurd. Thus the function is surjective, and so \( f \) is a bijection. \[ \square \]

**Theorem 5.8.** Suppose \( A \) is a set such that there exists a surjection \( \mathbb{N} \to A \). Then \( A \) is countable.

Proof. (Left as an exercise.) \[ \square \]

**Theorem 5.9.** \( \mathbb{N} \sim \mathbb{N} \times \mathbb{N} \).

Proof. (Left as an exercise.) (Hint: We can arrange the elements of \( \mathbb{N} \times \mathbb{N} \) in an array

\[
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

These can be ordered by going along the diagonals for which the sum of the two coordinates are constant:

\[
(1,1); (2,1), (1,2); (3,1), (2,2), (1,3); \ldots
\]

\[ \square \]

**Lemma 5.10.** If \( A_1 \sim B_1 \) and \( A_2 \sim B_2 \), then \( A_1 \times A_2 \sim B_1 \times B_2 \).

Proof. (Left as an exercise.) \[ \square \]

**Proposition 5.11.** \( \mathbb{N} \sim \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \).

Proof. (Left as an exercise.) (Hint: It may be beneficial to prove first that \( \mathbb{Z} \sim \mathbb{Z} \setminus \{0\} \).) \[ \square \]
Theorem 5.12. The set $\mathbb{Q}$ is countable.

Proof. (Left as an exercise.) \qed