1. Let \((X,d)\) be a metric space and let \(A \subseteq X\). Prove that 
\[ \overline{A} = A \cup A'. \]

2. Let \((X,d)\) be a metric space, let \(N \in \mathbb{N}\), and let \(A,A_j,A_{\alpha} \subseteq X\) for \(1 \leq j \leq N\) and all \(\alpha\) in some index set \(I\). You are asked to prove several topological facts below:
   (a) Prove that if \(A_1 \subseteq A_2\), then 
   \[ A_1^c \subseteq A_2^c \quad \text{and} \quad \overline{A_1} \subseteq \overline{A_2}. \]
   Give an example where \(A_1 \nsubseteq A_2\) and yet we have both of the equalities 
   \[ A_1^c = A_2^c \quad \text{and} \quad \overline{A_1} = \overline{A_2}. \]
   
   (b) Prove that 
   \[ \bigcup_{\alpha \in I} A_{\alpha} \subseteq \left( \bigcup_{\alpha \in I} A_{\alpha} \right)^c, \quad \bigcap_{j=1}^N A_j \subseteq \left( \bigcap_{j=1}^N A_j \right)^c, \quad \text{and} \quad \bigcap_{\alpha \in I} A_{\alpha} \subseteq \bigcap_{\alpha \in I} A_{\alpha}^c. \]
   Give an example of a finite collection of \(A_{\alpha}\) where the first set inequality is strict, and give an example where the second set inequality is strict.
   
   (c) Prove that 
   \[ \bigcup_{j=1}^N A_j' = \left( \bigcup_{j=1}^N A_j \right)' \quad \text{and} \quad \bigcup_{\alpha \in I} A_{\alpha}' \subseteq \left( \bigcup_{\alpha \in I} A_{\alpha} \right)' \quad \text{and} \quad \left( \bigcap_{\alpha \in I} A_{\alpha} \right)' \subseteq \bigcap_{\alpha \in I} A_{\alpha}'. \]
   Give an example where the first set inequality is strict, and give an example of a finite collection of \(A_{\alpha}\) where the second set inequality is strict.
   
   (d) Prove that 
   \[ \bigcup_{j=1}^N \overline{A_j} = \bigcup_{j=1}^N A_j, \quad \bigcup_{\alpha \in I} \overline{A_{\alpha}} \subseteq \bigcup_{\alpha \in I} A_{\alpha}, \quad \text{and} \quad \bigcap_{\alpha \in I} A_{\alpha} \subseteq \bigcap_{\alpha \in I} \overline{A_{\alpha}}. \]
   Give an example where the first set inequality is strict, and give an example of a finite collection of \(A_{\alpha}\) where the second set inequality is strict.
   (Hint: You may find it useful to use your answer to Problem #1 in conjunction with part (c) of this problem.)
   
   (e) Prove that 
   \[ \text{Fr} A = [A \cap (X\setminus A)]' \cup [(X\setminus A) \cap A']. \]
   
   (f) Prove also that 
   \[ \text{Fr} A = \overline{A} \cap \overline{X\setminus A}. \]
   (Hint: Use Problem #1 and part (e) of this problem.)
(g) Let Is A denote the set of isolated points of A. Prove that
\[ \text{Is } A = A \setminus A'. \]

(h) Now prove that one can choose \((X, d)\) and \(A\) such that \(A = \text{Is } A\) but yet \(A\) is not closed.

(Please note: This problem had previously asked you to prove that if \(A = \text{Is } A\), then \(A\) is closed, but it is not true!)

3. Write a careful proof that, in \(\mathbb{R}\),
\[ Q^c = \emptyset \quad \text{and} \quad \overline{Q} = \mathbb{R}. \]
Now prove that if \(I\) is an interval in \(\mathbb{R}\) (open or closed, finite or infinite), then
\[ (I \cap \mathbb{Q})^c = \emptyset \quad \text{and} \quad \overline{I \cap \mathbb{Q}} = \overline{I}. \]

4. For \(j \in \mathbb{N}\), let \((X_j, d_j)\) be a metric space. Suppose that the metrics \(d_j\) are uniformly bounded, i.e.,
\[ \exists M \in \mathbb{R} \text{ s.t. } \forall j \in \mathbb{N}, \quad \forall x, y \in X_j, \quad d_j(x, y) \leq M. \]
Set \(X := X_1 \times X_2 \times \cdots\). Define the function \(d : X \times X \to \mathbb{R}\) by
\[ d(x, y) := \sum_{j=1}^{\infty} \alpha_j d_j(x_j, y_j), \]
where \(x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)\), each \(\alpha_j > 0\), and the series \(\sum_{j=1}^{\infty} \alpha_j\) is convergent.

(a) Prove that \((X, d)\) is a metric space.
(b) Prove that the metric \(d\) is bounded, i.e.,
\[ \exists M' \in \mathbb{R} \text{ s.t. } \forall x, y \in X, \quad d(x, y) \leq M'. \]
We will consider this space further in the next problem set.

5. Let \((X, d)\) be a complete metric space and let \(Y \subseteq X\). Give \(Y\) the natural metric, \(\rho := d|_{Y \times Y}\), that it inherits as a subset of \(X\). Prove that \((Y, \rho)\) is a complete metric space if and only if \(Y\) is a closed subset of \(X\).

6. On Problem Set #4, you proved that if \(n \in \mathbb{N}\) and \(1 \leq p \leq q \leq \infty\), then \(\overline{B_{1, p, n}}(0) \subseteq \overline{B_{1, q, n}}(0)\).

(a) Prove that, given \(n \in \mathbb{N}\) and \(1 \leq p \leq q \leq \infty\), there is a constant \(c\) (which may depend upon \(n, p, \) and \(q\)) such that, for all \(x \in \mathbb{R}^n\),
\[ c \|x\|_p \leq \|x\|_q \leq \|x\|_p. \]
(b) Now prove that \(c\) can be replaced with \(C\), which depends upon \(n\), but not upon \(p\) or \(q\).
(c) Conclude that, if \(n \in \mathbb{N}\) and \(p \in [1, \infty]\) is fixed, a sequence \((x_j)\) in \((\mathbb{R}^n, d_p)\) converges if and only if the same sequence converges in \((\mathbb{R}^n, d_q)\) for all \(q \in [1, \infty]\).

Hint: For (a), you may wish to treat the three cases (i) \(p \leq q < \infty\), (ii) \(p < q = \infty\), and (iii) \(p = q = \infty\) separately; Hölder’s inequality may come in handy for case (i); for (b), try to find the choice of \(p\) and \(q\) that minimizes your answer to (a).

7. For each \(n \in \mathbb{N}\), define
\[ U_n := \{ (x, y) \in \mathbb{R}^2 : d((x, y), (0, n)) < n \}, \]
where \(d = d_2\) is the usual metric on \(\mathbb{R}^2\). Prove that
\[ \bigcup_{n=1}^{\infty} U_n = \mathbb{R} \times (0, \infty). \]
8. Let \((X, d)\) be a metric space and let \(A \subseteq X\) be a nonempty closed set. Fix \(x \in A\). What must be true about \(x\) in order that \(A \setminus \{x\}\) is also closed?

9. Let \((X, d)\) and \((Y, \rho)\) be metric spaces. Give the set \(X \times Y\) the metric \(\eta\) defined by

\[
\eta((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2).
\]

Prove that \((X \times Y, \eta)\) is a complete metric space if and only if \((X, d)\) and \((Y, \rho)\) are complete metric spaces.

10. Let \((X, d)\) be a metric space. If \(A\) and \(B\) are nonempty subsets of \(X\) and \(x \in X\), we define the distance from the point \(x\) to the set \(A\) and the distance from the set \(A\) to the set \(B\) as follows:

\[
d(x, A) := \inf \{d(x, a) : a \in A\} \quad \text{and} \quad d(A, B) := \inf \{d(a, b) : (a, b) \in A \times B\}.
\]

It is a fact that we will prove later (when we have the tools) that, if \(A\) is compact, then there is an \(a \in A\) such that \(d(x, a) = d(x, A)\). For now, content yourself with the following:

(a) Show by example that if \(A\) is not closed, there may not be such an \(a \in A\).

(b) Show by example that, even if \(A\) and \(B\) are closed, it may be the case that there is no \((a, b) \in A \times B\) such that \(d(a, b) = d(A, B)\).