1. Let \((X, d)\) be a metric space. Prove that \((\overline{d}, d)\), where \(\overline{d} : X \times X \to \mathbb{R}\) is defined by

\[\overline{d}(x, y) := \min\{d(x, y), 1\}\]

is also a metric space.

2. Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces. Set \(X := X_1 \times X_2\). Define the function \(d : X \times X \to \mathbb{R}\) by

\[d(x, y) := d_1(x_1, y_1) + d_2(x_2, y_2),\]

where \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\). Prove that \((X, d)\) is a metric space.

3. Generalize Problem #2. For \(1 \leq j \leq N\), let \((X_j, d_j)\) be a metric space. Set \(X := X_1 \times X_2 \times \cdots \times X_N\). Define the function \(d : X \times X \to \mathbb{R}\) by

\[d(x, y) := \sum_{j=1}^{N} \alpha_j d_j(x_j, y_j),\]

where \(x = (x_1, x_2, \ldots, x_N)\), \(y = (y_1, y_2, \ldots, y_N)\), and each \(\alpha_j > 0\). Prove that \((X, d)\) is a metric space. What happens if one of the \(\alpha_j \leq 0\)?

4. (a) Try to generalize Problem #3. For \(j \geq 1\), let \((X_j, d_j)\) be a metric space. Set \(X := X_1 \times X_2 \times \cdots\). Define the function \(d : X \times X \to \mathbb{R}\) by

\[d(x, y) := \sum_{j=1}^{\infty} \alpha_j d_j(x_j, y_j),\]

where \(x = (x_1, x_2, \ldots)\), \(y = (y_1, y_2, \ldots)\), and each \(\alpha_j > 0\). What problems might one encounter in trying to prove that \((X, d)\) is a metric space?

(b) Consider the concrete case where each \(X_j = \mathbb{R}\), each \(\alpha_j = 1\), and each \(d_j\) is the usual metric (i.e., the absolute-value function) on \(\mathbb{R}\). Investigate the problems you wrote about abstractly in part (a) in this specific case.

(c) Continuing with the hypotheses of part (b), how might you modify the set \(X\) or the function \(d\) in order to make \(X\) a metric space? Next week’s problem set will investigate metrics on this space.

5. Let \(X\) be a vector space over the real numbers. We say that a function \(n : X \to \mathbb{R}\) is a norm on \(X\) if \(n\) satisfies the following for all \(x, y \in X\) and all \(\alpha \in \mathbb{R}\):

(a) \(n(x) \geq 0\);
(b) \(n(x) = 0 \iff x = 0\);
(c) \(n(\alpha x) = |\alpha| n(x)\);
(d) \(n(x + y) \leq n(x) + n(y)\).

If \(n\) is a norm on \(X\), we often write \(n(x)\) as \(\|x\|\) and we call the pair \((X, \|\cdot\|)\) a normed space.

Let \((X, \|\cdot\|)\) be a normed space, and define \(d : X \times X \to \mathbb{R}\) by \(d(x, y) := \|x - y\|\). Prove that \((X, d)\) is a metric space. (The conclusion of this problem is that all normed spaces can be regarded as metric spaces in a very natural way.)
6. Let $X$ be a vector space over the real numbers. We say that a function $i : X \times X \to \mathbb{R}$ is an inner product on $X$ if $i$ satisfies the following for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{R}$:

(a) $i(\alpha x + \beta y, z) = \alpha i(x, z) + \beta i(y, z)$;
(b) $i(y, x) = i(x, y)$;
(c) $i(x, x) \geq 0$;
(d) $i(x, x) = 0 \iff x = 0$.

If $i$ is an inner product on $X$, we often write $i(x, y)$ as $\langle x, y \rangle$ and we call the pair $(X, \langle \cdot, \cdot \rangle)$ an inner-product space.

Prove that if $(X, \langle \cdot, \cdot \rangle)$ is an inner-product space, then, for all $x, y \in X$,

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}},$$

by using the following outline:

(I) For fixed $\lambda \in \mathbb{R}$, consider the quantity $\langle x - \lambda y, x - \lambda y \rangle$ and prove that

$$\langle y, y \rangle \lambda^2 - 2 \langle x, y \rangle \lambda + \langle x, x \rangle \geq 0.$$

(II) Letting $a = \langle y, y \rangle$, $b = -2 \langle x, y \rangle$, and $c = \langle x, x \rangle$, the above describes a quadratic function $a \lambda^2 + b \lambda + c$ of $\lambda$ that is always nonnegative. What can you say about the relationship between $a$, $b$, and $c$, for such a function?

(III) Translate your answer to (II) back into the language of the inner product to obtain the desired inequality.

This is one form of the Cauchy-Schwarz inequality. When $X = \mathbb{R}^n$ and the inner product is the usual dot product in $\mathbb{R}^n$, then this inequality is equivalent to the $p = 2 = q$ case of Hölder’s inequality that we proved in class. It is not a controversial statement to say that it is one of the most important inequalities in all of mathematics, so learn it well.

7. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner-product space, and define $d : X \times X \to \mathbb{R}$ by $d(x, y) := \langle x - y, x - y \rangle^{\frac{1}{2}}$.

Prove that $(X, d)$ is a metric space. You may want to do this by first proving that $(X, \|\cdot\|)$ is a normed space, where $\|\cdot\| : X \to \mathbb{R}$ is defined by $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$ and then appealing to your result from Problem #5. Along the way, you should also have to use your result from Problem #6 in order to prove that your prospective norm satisfies the triangle inequality. (The conclusion of this problem is that all inner-product spaces can be regarded as normed spaces and metric spaces in a very natural way.)

8. In class we have defined norms $\|\cdot\|_p$ on $\mathbb{R}^n$ for $p \in [1, \infty]$. In this problem, some of their relationships are investigated.

(a) Prove that if $n = 1$, then all of the $\|\cdot\|_p$ are the same.

(b) For $n = 2$, draw the sets $B_{1,p,n}(0)$ defined by

$$B_{1,p,n}(0) := \left\{ x \in \mathbb{R}^n : \|x\|_p \leq 1 \right\}$$

for $p = 1$, $p = 2$, and $p = \infty$ on the same set of axes. What is the containment relationship between them?

(c) Now prove that, for any $n \in \mathbb{N}$, if $1 \leq p \leq q \leq \infty$, then

$$B_{1,p,n}(0) \subseteq B_{1,q,n}(0).$$