POLYNOMIALS WITH SPECIFIED ROOT MULTIPLICITIES

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Abstract. Fix a partition $\lambda$ of $n > 0$. The space of monic polynomials of degree $n$ whose root multiplicities partition $n$ by $\lambda$ forms an algebraic variety. We give a method for computing its class in the Grothendieck ring of varieties and use the Grothendieck-Lefschetz Trace Formula to count the number of such polynomials over a finite field.

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0. Introduction

A monic polynomial $f(z)$ of degree $n > 0$ with coefficients in a field $K$ and $m \leq n$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ in an algebraic closure $\bar{K}$ can be factored as

\[(z - \alpha_1)^{\lambda_1} (z - \alpha_2)^{\lambda_2} \cdots (z - \alpha_m)^{\lambda_m}\]

for some positive integers $\lambda_i$ which partition $n$. We say $\lambda_i$ is the multiplicity of the root $\alpha_i$.

Definition. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of $n$, define $Poly_n(\lambda)$ as the set of such $f(z)$ yielding $\lambda$ as above.

An important special case is when $\lambda = 1 + 1 + \ldots + 1$. $Poly_n(1 + 1 + \ldots + 1)$ is the space of monic square-free polynomials, and we write this simply as $Poly_n$.

There may be repetitions among the multiplicities $\lambda_i$. Let $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}$ be the set of distinct multiplicities obtained by forgetting these repetitions. Write $\tau_j$ for the number of times $\lambda_{i_j}$ appears in $\lambda$, i.e. the number of distinct roots of $f(z)$ which are assigned multiplicity $\lambda_{i_j}$. The partition $\tau_1 + \tau_2 + \ldots + \tau_r = m$, denoted $\tau$, may also be obtained as follows. Assuming that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$, rewrite this as

$\lambda_1 = \lambda_2 \ldots = \lambda_{i_0} < \lambda_{i_1+1} = \lambda_{i_1+2} = \ldots = \lambda_{i_2} < \lambda_{i_2+1} = \ldots = \lambda_{i_r}$

where $0 = i_0 < i_1 < i_2 < \ldots < i_r = m$. Then $\tau_j = i_j - i_{j-1}$.

A key observation is that the partition $\tau$ is more essential to the structure of $Poly_n(\lambda)$ than $\lambda$, since an element of $Poly_n(\lambda)$ is uniquely characterized by a set of $m$ distinct roots in $\bar{K}$ partitioned, according to multiplicity, into $r$ labelled subsets.
with sizes given by $\tau$. For example, $Poly_{8}(1+2+2)$ is in bijection with $Poly_{11}(3+4+4)$. The relevant feature here is that a polynomial in either space has three distinct roots, two of which are labelled by a certain multiplicity and a third labelled by another, different multiplicity. The particular multiplicities chosen are irrelevant. In §1 we make this observation more precise, showing in Proposition 1.4 that $\tau$ determines the isomorphism class of $Poly_{n}(\lambda)$ as an algebraic variety.

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1. Poly$_n(\lambda)$ as a Variety

Any monic polynomial over $K$ of degree $n$ can be identified with a point of $K^n$ with coordinates given by its coefficients, so we can view affine $n$-space over $K$ as the coefficient space of such polynomials, and write this as $A_n^{coeff}$.

1.1. Proposition. (1) Poly$_n(\lambda)$ is a locally closed subvariety of $A_n^{coeff}$

(2) Poly$_n := Poly_n(1 + 1 + \ldots + 1)$ is an open subvariety of $A_n^{coeff}$

Proof. Affine $n$-space $K^n_{root} = \text{Spec } K[x_1, x_2, \ldots, x_n]$, thought of as the ordered space of roots of degree $n$ polynomials, maps to $A_n^{coeff}$ via the morphism of schemes $\pi : K^n_{root} \to A_n^{coeff}$ given by the elementary symmetric polynomials. $\pi$ is the quotient map by the action of $S_n$ on $K^n_{root}$ and is surjective on closed points.

Let $L_E$ be the $S_n$-orbit of the intersection of hyperplanes in $K^n_{root}$ defined by the equations

\begin{equation}
E := \{(i, j) : \sum_{\ell=1}^{k} \lambda_{\ell} < i, j \leq \sum_{\ell=1}^{k+1} \lambda_{\ell}, 1 \leq k \leq m\}
\end{equation}

which constrains the coordinates of $K^n_{root}$ to $m$ collections of $\lambda_k$ equal coordinates. The $K$-points of the image $\pi(L_E)$ of $L_E$ contains the set $\text{Poly}_n(\lambda)$ along with all polynomials with root multiplicities partitioning $n$ more coarsely than $\lambda$.

To cut $\pi(L_E)$ down to size, we remove polynomials with root multiplicities too coarse as follows. For each subset $E \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ strictly containing $E$, let $L_E$ be the intersection of hyperplanes defined as is $L_E$ by the set of equations \((1.1)\) above, but with further equations imposed by pairs of indices in $\hat{E}$.

Then

\begin{equation}
\pi(L_E) = \bigcup_{E \supseteq \hat{E}} \pi(L_{\hat{E}})
\end{equation}

has $K$-points Poly$_n(\lambda)$ and is a locally closed subvariety of $A_n^{coeff}$ because $\pi$ is a finite morphism (hence is closed) and all $L_E$ and $L_{\hat{E}}$ are closed in $K^n_{root}$, thus proving (1). For (2), note that if $\lambda = 1 + 1 + \ldots + 1$ then the equations given by $E$ impose no conditions on $K^n_{coeff}$ so that $\pi(L_E) = A_n^{coeff}$ and the subvariety given by \((1.2)\) above is open. □
Definition. \( \text{Conf}_m \), the ordered space of \( m \) distinct roots, denotes the \( K \)-subvariety of affine \( m \)-space \( \mathbb{A}_m^m \) given by \( x_i \neq x_j \) for \( i \neq j \).

When \( K = \mathbb{C} \), \( \text{Conf}_m \) is the configuration space of \( m \) labeled points in the complex plane, hence our notation.

The product of symmetric groups \( S_{\tau_1} \times S_{\tau_2} \times \ldots \times S_{\tau_r} \) acts on \( \text{Conf}_m \) by permuting coordinates. More precisely, \( \text{Conf}_m = \text{Spec} \left( S^{-1}K[x_1, x_2, \ldots, x_m] \right) \) where \( S^{-1}K[x_1, x_2, \ldots, x_m] \) denotes the localization of \( K[x_1, x_2, \ldots, x_m] \) at the multiplicative subset \( S \) generated by all \( x_i - x_j \) with \( i < j \). \( \prod_{j=1}^r S_{\tau_j} \) acts on \( K[x_1, x_2, \ldots, x_m] \) via the action of \( S_{\tau_j} \) on
\[
\left\{ x_{i_{j-1}+1}, x_{i_{j-1}+2}, \ldots, x_{i_j} \right\}
\]
given by
\[
\sigma_j(x_{i_{j-1}+k}) = x_{i_{j-1}+\sigma_j^{-1}(k)}
\]
for \( \sigma_j \in S_{\tau_j} \). Equivalently, this is the action obtained by viewing \( S_{\tau} \) inside the symmetric group \( S_m \) and restricting the action
\[
\sigma(x_j) = x_{\sigma^{-1}(j)}
\]

Notation. Write \( S_{\tau} \) for the product of symmetric groups \( \prod_{j=1}^r S_{\tau_j} \).

Since \( S_{\tau} \) maps \( S \) to itself, the action of \( S_{\tau} \) on \( K[x_1, x_2, \ldots, x_m] \) induces an action on the localization \( S^{-1}K[x_1, x_2, \ldots, x_m] \), hence on \( \text{Conf}_m \). The scheme-theoretic quotient by this action is the affine \( K \)-variety
\[
\text{Conf}_m/S_{\tau} = \text{Spec} \left( S^{-1}K[x_1, x_2, \ldots, x_m] \right)^{S_{\tau}}
\]
where \( (S^{-1}K[x_1, x_2, \ldots, x_m])^{S_{\tau}} \) denotes the subring of \( S^{-1}K[x_1, x_2, \ldots, x_m] \) invariant under \( S_{\tau} \).

We will show that \( \text{Conf}_m/S_{\tau} \cong \text{Poly}_n(\lambda) \) as \( K \)-varieties if \( K \) is perfect. Over \( \mathbb{C} \), this variety is a configuration space of \( m \) distinct points in the complex plane which have been ‘colored’ according to their multiplicities. It will turn out that this description is more useful for proving results about \( \text{Poly}_n(\lambda) \) than that given in Proposition 1.1 above.

Notation. Write \( A \) for the invariant subring \( K[x_1, x_2, \ldots, x_m]^{S_{\tau}} \), and \( B \) for \( (S^{-1}K[x_1, x_2, \ldots, x_m])^{S_{\tau}} \) so that \( \text{Conf}_m/S_{\tau} = \text{Spec} \ B \).

For \( j = 1, 2, \ldots, r \) and \( i = 1, 2, \ldots, \tau_j \), let \( \psi_j^i \) be the coefficient of \( z^{i-1} \) of the polynomial
\[
\prod_{k=1}^{\tau_j} (z - x_{i_{j-1}+k})
\]
in \( z \). \( \psi_j^i \) is a symmetric polynomial in \( x_{i_{j-1}+1}, x_{i_{j-1}+2}, \ldots, x_{i_j} \), hence belongs in \( A \).

1.2. Lemma. \( A \) is generated as a \( K \)-algebra by the collection of \( \psi_j^i \).

Proof. In fact the result is true if \( K \) is only a commutative ring; upon this hypothesis we induct on \( r \). When \( r = 1 \), \( A \) is just \( K[x_1, x_2, \ldots, x_{\tau_1}]^{S_{\tau_1}} \) and the result follows from the well known theorem on symmetric polynomials.

If \( r > 1 \), we have
\[
A = A'[x_{i_1+1}, x_{i_1+2}, \ldots, x_m] \prod_{j=1}^r S_{\tau_j}
\]
where \( A' = K[x_1, x_2, \ldots, x_{\tau_1}]^{S_{\tau_1}} \). The lemma follows from the inductive hypothesis. \( \square \)
1.3. **Lemma.** The map \((S \cap A)^{-1}A \hookrightarrow B\) induced by the injection \(A \hookrightarrow B\) is an isomorphism. In particular, every element of \(B\) can be written as a ratio of elements of \(A\).

**Proof.** Write an element of \(B\) in the form \(g/s\) where \(g \in K[x_1, x_2, \ldots, x_m]\) and \(s \in S\) with

\[ s = \prod_{i<j} (x_i - x_j)^{e_{ij}} \]

for some \(e_{ij} \geq 0\) and

\[ \left(\frac{g}{s}\right)^\sigma = \frac{g}{s} \]

for all \(\sigma \in S_\tau\).

For any transposition \((ij) \in S_\tau\), we have

\[ s^{(ij)} = (-1)^{e_{ij}}s \]

and thus

\[ \frac{g}{s} = \left(\frac{g}{s}\right)^{(ij)} = \frac{g^{(ij)}}{s^{(ij)}} = (-1)^{e_{ij}} \frac{g^{(ij)}}{s} \]

\[ \Rightarrow g^{(ij)} = (-1)^{e_{ij}}g \]

Therefore \(sg\) and \(s^2\) are fixed by all transpositions in \(S_\tau\). So \(sg \in A\) and \(s^2 \in A \cap S\), and \(g/s = \frac{sg}{2}\) lies in the image of \((S \cap A)^{-1}A\). \(\square\)

1.4. **Proposition.** If \(K\) is perfect then \(\text{Conf}_m/S_\tau\) and \(\text{Poly}_n(\lambda)\) are isomorphic as varieties over \(K\).

**Proof.** We show only that there is a morphism

\[ \iota : \text{Conf}_m/S_\tau \to \text{Poly}_n(\lambda) \]

of \(K\)-varieties which is bijective on \(K\)-points. An inverse can be constructed by mapping from the space of roots of \(\text{Poly}_n(\lambda)\).

For \(i = 1, 2, \ldots, n\) let \(\phi_i\) be the polynomial in \(x_1, x_2, \ldots, x_m\) given by the coefficient of \(z^{i-1}\) of the polynomial

\[ \prod_{j=1}^m (z - x_j)^{\lambda_j} \]

in \(z\) so that

\[ z^n + \phi_n z^{n-1} + \phi_{n-1} z^{n-2} + \ldots + \phi_2 z + \phi_1 = \prod_{j=1}^m (z - x_j)^{\lambda_j} = \prod_{j=1}^r \left( \prod_{k=1}^{\tau_j} (z - x_{ij-1+k}) \right)^{\lambda_j} \]

\[ = \prod_{j=1}^r \left( z^{\tau_j} + \psi_j^{\tau_j} z^{\tau_j-1} + \ldots + \psi_j^{1} z + \psi_j^{0} \right)^{\lambda_j} \]

(1.3)

Write \(\iota^f : K[y_1, y_2, \ldots, y_n] \to B\) for the map given by

\[ y_i \mapsto \phi_i \]

and let \(\iota : \text{Conf}_m/S_\tau \to A^n\) be the induced map on spectra. A \(K\)-point of \(\text{Conf}_m/S_\tau\) is a map \(\kappa\) of \(K\)-algebras

\[ \kappa : B \to K \]
The pullback $\kappa \circ \iota^z$ of $\kappa$ by $\iota^z$ is the map $y_i \mapsto \kappa(\phi_i)$ which can be identified with the polynomial

$$f(z) := z^n + \kappa(\phi_n)z^{n-1} + \kappa(\phi_{n-1})z^{n-2} + \ldots + \kappa(\phi_2)z + \kappa(\phi_1)$$

We show that the polynomial $\prod_{j=1}^rf_j$ where

$$f_j = z^{\tau_j} + \kappa(\psi_j^q)z^{\tau_j-1} + \ldots + \kappa(\psi_j^1)z + \kappa(\psi_j^0)$$

is square-free, which by equation (1.3) implies that $f(z) \in Poly_n(\lambda)$ and hence that $\iota$ maps $K$-points into $Poly_n(\lambda)$.

Let $\Delta$ be the discriminant of degree $m$. For $k = 1, 2, \ldots, m$ write $\Psi_k$ for the symmetric polynomial in $x_1, x_2, \ldots, x_m$ given by the coefficient of $z^{k-1}$ of the polynomial

$$\prod_{j=1}^m(z - x_j)$$

in $z$ so that

$$\Delta(\Psi_k) = \prod_{1 \leq i < j \leq m}(x_i - x_j)^2$$

Then

$$\prod_{j=1}^rf_j = z^m + \kappa(\Psi_m)z^{m-1} + \ldots + \kappa(\Psi_2)z + \kappa(\Psi_1)$$

and

$$\Delta(\kappa(\Psi_k)) = \kappa(\Delta(\Psi_k)) \neq 0$$

since $\Delta(\Psi_k) \in S$ by equation (1.4) above and $\kappa$ does not vanish on the localized set $S$. Thus $\prod_{j=1}^rf_j$ is square-free and $\iota(\kappa) \in Poly_n(\lambda)$.

Any $f(z) \in Poly_n(\lambda)$ can be factored as

$$f(z) = \prod_{j=1}^rf_j^{\lambda_j}$$

where each $f_j$ is the unique monic, square-free polynomial of degree $\tau_j$ whose roots are exactly the distinct roots of $f$ with multiplicity $\lambda_j$. A priori, $f_j$ is only contained in $\bar{K}[z]$, but in fact it has coefficients in $K$: if $\sigma \in Gal(K/K)$ then

$$f(z) = f(z)^\sigma = \prod_{j=1}^r(f_j^\sigma)^{\lambda_j}$$

so that $f_j = f_j^\sigma$ and $f_j \in K[z]$.

Since the $f_j$ are uniquely determined by $f(z)$, the values of $\kappa$ on $\psi_j$ are uniquely determined by its values on $\phi_i$ due to equation (1.3) above. By Lemmas 1.2 and 1.3 $\kappa$ is uniquely determined by its values on $\psi_j$. Furthermore, any $K$-point of $f(z) \in Poly_n(\lambda)$ arises as the pull-back under $\iota$ of $\kappa$ mapping $\psi_j$ to the coefficient of $z^{j-1}$ in $f_j$. It follows that $\iota$ is injective on $K$-points and surjects onto $Poly_n(\lambda)$. □
Convention. In light of Proposition 1.4, we will assume $K$ is perfect for the remainder of this paper.

1.5. Proposition. $\text{Poly}_n(\lambda)$ embeds as an open subvariety, written $\text{Poly}_\tau$, of the product of square-free polynomial spaces

\[ \text{Poly}_{\tau_1} \times \text{Poly}_{\tau_2} \times \ldots \times \text{Poly}_{\tau_r} \]

1.5.1. Corollary. Over a finite field $\mathbb{F}_q$ of order $q$,

\[ |\text{Poly}_n(\lambda)/\mathbb{F}_q| = \sum_{g \in \text{Poly}_m/\mathbb{F}_q} |\nu^{-1}(g)| \]

where $\nu : \text{Poly}_\tau \to \text{Poly}_m$ is the multiplication map

\[ (f_j) \mapsto \prod f_j \]

and $|\nu^{-1}(g)|$ is the size of the fiber over a square-free polynomial $g$, i.e., the number of ordered lists of factors $(f_1, f_2, \ldots, f_r)$ of $g$ with degrees specified by $\tau$ multiplying to $g$.

2. Preliminaries

We recall two notions from algebraic geometry which will be used to count $\text{Poly}_n(\lambda)$ over a finite field.

The Grothendieck-Lefschetz Trace Formula. For a smooth projective variety $X$ over $\mathbb{F}_q$ the number $|X(\mathbb{F}_q)|$ of $\mathbb{F}_q$-points is given by the formula

\[ |X(\mathbb{F}_q)| = \sum_{i \geq 0} (-1)^i \text{Trace}(\text{Frob}_q^* | H_{\ell}^i(X; \mathbb{Q}_\ell)) \]

where $\ell$ is a prime different from the characteristic of $\mathbb{F}_q$, $X$ is the $\overline{\mathbb{F}}_q$-variety $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and $\text{Frob}_q^* | H_{\ell}^i(\overline{X}; \mathbb{Q}_\ell)$ is the map on the $i^{th}$ $\ell$-adic cohomology group induced by the geometric Frobenius morphism $\text{Frob}_q : \overline{X} \to \overline{X}$.

The trace formula [2.1] holds for non-projective $X$ if we replace $H_{\ell}^i(\overline{X}; \mathbb{Q}_\ell)$ by compactly supported étale cohomology groups $H_{\ell, c}^i(\overline{X}; \mathbb{Q}_\ell)$ (cf. [2]), so that

\[ |X(\mathbb{F}_q)| = \sum_{i \geq 0} (-1)^i \text{Trace}(\text{Frob}_q^* | H_{\ell, c}^i(\overline{X}; \mathbb{Q}_\ell)) \]

Furthermore, if $X$ is smooth, then Poincaré duality for étale cohomology ([4], Theorem 24.1) gives an isomorphism of Galois representations

\[ H_{\ell, c}^i(\overline{X}; \mathbb{Q}_\ell) \cong H_{\ell}^{2\dim X-i}(\overline{X}; \mathbb{Q}_\ell(-\dim X))^{\vee} \]
where, for an integer \( e \), \( Q_\ell(e) \) denotes the 1-dimensional Galois representation over \( \mathbb{Q}_\ell \) on which \( \text{Frob}_q \) acts with weight \( q^{-e} \). Plugging this into formula (2.2), we obtain

\[
|X(\mathbb{F}_q)| = q^{\dim X} \sum_{i \geq 0} (-1)^i \text{Trace}(\text{Frob}_q^* | H^i_\text{dR}(X; \mathbb{Q}_\ell)^\vee) \tag{2.3}
\]

The Grothendieck ring of varieties.

**Definition.** The Grothendieck ring \( K_0(\text{Var}_K) \) of varieties over \( K \) is, as a group, the quotient of the free abelian group on isomorphism classes \([X]\) of \( K \)-varieties \( X \) by the relation

\[ [X] = [X - Z] + [Z] \]

for a closed subvariety \( Z \subset X \). Multiplication is given by

\[ [X][Y] = [X \times_K Y] \]

We denote the class \([\mathbb{A}^1]\) of the affine line by \( L \).

In §3 we give an algorithm to compute the class of \( \text{Poly}_n(\lambda) \) in the Grothendieck ring. The following lemma implies that the algorithm, in particular, counts \( \text{Poly}_n(\lambda) \) over a finite field:

**2.1. Lemma.** The map \( K_0(\text{Var}_{\mathbb{F}_q}) \to \mathbb{Z} \) which sends the class \([X]\) of an \( \mathbb{F}_q \)-variety \( X \) to the number of \( \mathbb{F}_q \)-points of \( X \) is a ring homomorphism. \( \square \)

We will also use the following crucial fact about \( K_0(\text{Var}_K) \):

**2.2. Lemma.** If \( X \) is a disjoint union of locally closed subvarieties \( X_1, X_2, \ldots, X_e \) then

\[ [X] = [X_1] + [X_2] + \ldots + [X_e] \]

**Proof.** See the proof of Lemma 2.2 in [1] \( \square \)

3. Calculations in the Grothendieck ring

As we will see in Theorem 3.2, the class of \( \text{Poly}_n(\lambda) \) in the Grothendieck ring is a polynomial in classes of square-free spaces \([\text{Poly}_d]\), so we first show that \([\text{Poly}_d]\) has a simple formula in terms of \( L \):

**3.1. Proposition.** \([\text{Poly}_n] = L^n - L^{n-1}\) in the Grothendieck ring \( K_0(\text{Var}_K) \) if \( n > 1 \)

**Proof.** We induct on \( n \). For \( \ell \) a nonnegative integer such that \( n - 2\ell \geq 0 \) let \( R_\ell \) be the subset of \( \mathbb{A}^n_{\text{coeff}} \) containing all polynomials which can be written in the form \( ab^2 \), where \( a \) and \( b \) are polynomials over \( K \) of degree \( n - 2\ell \) and \( \ell \) respectively. Then we have the filtration

\[ \emptyset = R_{d+1} \subset R_d \subset R_{d-1} \subset \ldots \subset R_1 \subset R_0 = \mathbb{A}^n \]

where \( d = \lfloor n/2 \rfloor \).

The closed subvariety of \( \mathbb{A}^n_{\text{root}} \) given by the \( S_n \)-orbit of the intersection of hyperplanes

\[ x_j = x_{j+1} \]

for \( j = 1, 3, 5, \ldots, 2\ell - 1 \) is mapped onto \( R_\ell \) by the quotient

\[ \mathbb{A}^n_{\text{root}} \overset{S_n}{\longrightarrow} \mathbb{A}^n_{\text{coeff}} \]

which is a closed map, so \( R_\ell \) is closed.
Writing $Z_{\ell}$ for the locally closed subvariety $R_{\ell} - R_{\ell+1}$, we have
\[ [\mathbb{A}^n] = [Z_0] + [Z_1] + [Z_2] + \ldots + [Z_d] \]

The morphism of varieties $\text{Poly}_{n-2\ell} \times \mathbb{A}^\ell \to Z_\ell$ which maps a squarefree $a$ of degree $n-2\ell$ and $b$ of degree $\ell$ by
\[ (a, b) \mapsto ab^2 \]
is in fact an isomorphism; one can construct its inverse by mapping from the space of roots of $Z_\ell$. Thus $Z_0 \cong \text{Poly}_n$ and
\[ [Z_d] = [\text{Poly}_{n-2d} \times \mathbb{A}^d] = \mathbb{L}^{n-d} \]
and by the inductive hypothesis
\[ [Z_\ell] = [\text{Poly}_{n-2\ell} \times \mathbb{A}^\ell] = (\mathbb{L}^{n-2\ell} - \mathbb{L}^{n-2\ell-1})\mathbb{L}^\ell = \mathbb{L}^{n-\ell} - \mathbb{L}^{n-\ell-1} \]
for $0 < \ell < d$ and so
\[ [\text{Poly}_n] = [Z_0] = [\mathbb{A}^n] = ([Z_1] + [Z_2] + \ldots + [Z_d]) = \mathbb{L}^n - \sum_{\ell=1}^{d-1} (\mathbb{L}^{n-\ell} - \mathbb{L}^{n-\ell-1}) - [Z_d] = \mathbb{L}^n - \mathbb{L}^{n-1} + \mathbb{L}^{n-d} - [Z_d] = \mathbb{L}^n - \mathbb{L}^{n-1}. \]

3.2. Theorem. There is a recursive formula, given by equation (3.3) below, for the class $[\text{Poly}_n(\lambda)] = [\text{Poly}_r]$ of $\text{Poly}_n(\lambda)$ in the Grothendieck ring which reduces to
\[ (3.1) \quad [\text{Poly}_n(\lambda)] = [\text{Poly}_r] = \prod_p (\text{Poly}_r) + \sum_{|\Lambda| < |r|} a_{\Lambda} \left( \prod_j [\text{Poly}_{\Lambda_j}] \right) \]
for some $a_{\Lambda} \in \mathbb{Z}$, where the sum on the righthand side is taken over all partitions $\Lambda$ of integers smaller than $|r| = \sum \tau_j$.

Proof. We induct on $m = |r|$, the case $m = 1$ being trivially true, so assume $m > 1$ and suppose $(f_1, f_2, \ldots, f_r) \in \prod \text{Poly}_r$. For each nonempty subset $S \subset \{1, 2, \ldots, r\}$, the set of $\alpha \in K$ such that
\begin{enumerate}
  \item $f_j(\alpha) = 0$ for $j \in S$
  \item $f_j(\alpha) \neq 0$ for $j \notin S$
\end{enumerate}
is finite and Galois-invariant, hence is precisely the set of roots of a unique monic square-free polynomial $h_S$ over $K$.

Alternatively, $h_S$ is the monic polynomial of largest degree dividing precisely those $f_k$ with $k \in S$. The roots of $h_S$ are the roots shared by all $f_k \in S$ and no other $f_j$.

Thus $(f_1, f_2, \ldots, f_r)$ induces a function, which we write as $F$, from the set $\mathcal{P}(r)$ of nonempty subsets of $\{1, 2, \ldots, r\}$ to non-negative integers $\mathbb{Z}_{\geq 0}$ given by
\[ F(S) = \deg(h_S) \]
which satisfies, for all $j$,
\[ \sum_{S \supset j} F(S) = \tau_j \]
where the sum is taken over $S$.

Roughly speaking, $F$ keeps track of the manner in which the $f_j$ are sharing roots. The equality (3.2) says that each of the $\tau_j$ roots of $f_j$ is shared via a unique $S \in \mathcal{P}(r)$.
**Notation.** We write

(1) \( \mathcal{F} \) for the set of all \( F : \mathcal{P}(r) \to \mathbb{Z}_{\geq 0} \) satisfying (3.2).

(2) \( Z_F \) for the subset of \( \prod Poly_{\tau_j} \) containing all \( (f_1, f_2, \ldots, f_r) \) inducing \( F \).

(3) \( F_0 \in \mathcal{F} \) for the map \( F_0(\{j\}) = \tau_j \) which is 0 elsewhere.

Note that \( (f_j) \) induces \( F_0 \) if and only if no roots are shared among \( (f_j) \), so that

\[ Z_{F_0} = Poly_{\tau}. \]

\( Z_F \) is locally closed in \( \prod Poly_{\tau_j} \) since it is the intersection of images, and complements thereof, under the quotient map

\[ \prod Conf_{\tau_j} \xrightarrow{S} \prod Poly_{\tau_j} \]

of finitely many closed subsets of \( \prod Conf_{\tau_j} \) given by intersections of hyperplanes (cf. proof of Proposition 1.1).

\( Z_F \) is isomorphic to \( Poly_{\tau(F)} \) (cf. Proposition 1.5), where \( \tau(F) \) is the partition

\[ \sum_{S \in \mathcal{P}(r)} F(S) \]

via the correspondence

\[ (h_S)_{S \in \mathcal{P}(r)} \leftrightarrow (\prod_{S \ni j} h_S)^{\tau_j} \]

which ‘extracts and collapses’ common divisors among \( (f_j) \) to obtain polynomials \( (h_S) \) which do not share roots. Since \( \prod Poly_{\tau_j} \) is a disjoint union of the \( Z_F \), it follows from Lemma 2.2 that

\[ \prod Poly_{\tau_j} = \sum_{F \in \mathcal{F}} [Z_F] = \sum_{F \in \mathcal{F}} [Poly_{\tau(F)}] \]

Solving for \( [Poly_n(\lambda)] \), we obtain

(3.3) \[ [Poly_n(\lambda)] = [Poly_{\tau}] = [Z_{F_0}] = \prod_{F \in \mathcal{F}} [Poly_{\tau_j}] - \sum_{F \in \mathcal{F}, F \neq F_0} [Poly_{\tau(F)}] \]

while \( |\tau(F)| < |\tau| \) for all \( F \neq F_0 \), so by the inductive hypothesis each \( [Poly_{\tau(F)}] \) satisfies the formula 3.1 above, which, plugged into equation 3.3 completes the proof.

\( \square \)

**4. Trace Formula**

The formula given in Proposition 3.1 and the counting homomorphism in Lemma 2.1 together imply that the number of monic, square-free polynomials of degree \( m \) over a finite field \( \mathbb{F}_q \) of order \( q \) is \( q^m - q^{m-1} \). In this section we obtain this count using the Grothendieck-Lefschetz Trace Formula, introduced in §2, and attempt to generalize the method to \( Poly_n(\lambda) \) for arbitrary \( \lambda. \)

Let \( Frob_q : Conf_{m/\mathbb{F}_q} \to Conf_{m/\mathbb{F}_q} \) be the geometric Frobenius automorphism which raises coordinates to the \( q^i \)th power. \( Frob_q \) acts on the \( i \)th \( \ell \)-adic cohomology group \( H^i_{\text{ét}}(Conf_{m/\mathbb{F}_q}; \mathbb{Q}_\ell) \) by multiplication by \( q^i \) (6, Proposition 3.3). \( Poly_{m/\mathbb{F}_q} \)

is the quotient of \( Conf_{m/\mathbb{F}_q} \) by \( S_m \), so transfer gives an isomorphism of Galois representations

\[ H^i_{\text{ét}}(Poly_{m/\mathbb{F}_q}; \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(Conf_{m/\mathbb{F}_q}; \mathbb{Q}_\ell)^{S_m}. \]
i.e. the subspace of $H^1_{\text{et}}(\text{Conf}_m/\mathbb{F}_q; \mathbb{Q}_\ell)$ fixed by the action of $S_m$, which is indeed a Galois representation because the actions of $S_m$ and Frobenius commute.

$\text{Poly}_m/\mathbb{F}_q$ is smooth and $m$-dimensional since it is an open subvariety of affine $m$-space, so the trace formula (2.3) gives

\begin{equation}
|\text{Poly}_m/\mathbb{F}_q| = q^m \sum_{i \geq 0} (-q)^{-i} \dim H^i_{\text{et}}(\text{Conf}_m/\mathbb{F}_q; \mathbb{Q}_\ell)^{S_m}
\end{equation}

It therefore suffices to find the dimension of $H^i_{\text{et}}(\text{Conf}_m/\mathbb{F}_q; \mathbb{Q}_\ell)^{S_m}$, which, by Artin’s comparison theorem, is the same as the $\mathbb{C}$-dimension of the analogous singular cohomology group $H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})^{S_m}$. The singular cohomology ring $H^*(\text{Conf}_m/\mathbb{C}; \mathbb{C})$ and its structure as an $S_m$-representation is well understood and was studied by Arnol’d (see [5] and §2.2 in [7]), whose results show that

\[ H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})^{S_m} \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 1 \\ 0 & \text{else} \end{cases} \]

From equation (4.1) it follows that

\[ |\text{Poly}_m/\mathbb{F}_q| = q^m (1 - q^{-1}) = q^m - q^{m-1} \]

Now we count $\text{Poly}_n(\lambda)$, which, as an open subvariety of $\prod \text{Poly}_{\mathbb{F}_q}$, is smooth and $m$-dimensional. As before, transfer gives an isomorphism

\[ H^i_{\text{et}}(\text{Poly}_n(\lambda)/\mathbb{F}_q; \mathbb{Q}_\ell) \cong H^i_{\text{et}}(\text{Conf}_m/\mathbb{F}_q; \mathbb{Q}_\ell)^{S_{\tau}} \]

of Galois representations and therefore

\begin{equation}
|\text{Poly}_n(\lambda)/\mathbb{F}_q| = q^m \sum_{i \geq 0} (-q)^{-i} \dim H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})^{S_{\tau}}
\end{equation}

By Frobenius reciprocity,

\begin{equation}
\dim H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})^{S_{\tau}} = \left\langle H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C}), 1 \right\rangle_{S_{\tau}}
\end{equation}

\begin{equation}
= \left\langle H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C}), \chi \right\rangle_{S_{\tau}}
\end{equation}

is an inner product of characters of representations of $S_m$, where $\chi$ denotes the character of the induced representation $\text{Ind}_{S_{\tau}}^{S_m}(1)$ of $S_m$ by the trivial representation of $S_{\tau}$.

Plugging into (4.2), we have

\begin{equation}
|\text{Poly}_n(\lambda)/\mathbb{F}_q| = q^m \sum_{i \geq 0} (-q)^{-i} \left\langle H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C}), \chi \right\rangle_{S_m}
\end{equation}

Church, Ellenberg, and Farb prove in Theorem 3.7 of [6], using a version of Grothendieck-Lefschetz with twisted coefficients, the following formula for any class function $\chi$ on $S_m$:

\[ q^m \sum_{i \geq 0} (-q)^i \left\langle H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C}), \chi \right\rangle_{S_m} = \sum_{g \in \text{Poly}_m/\mathbb{F}_q} \chi(\sigma_g) \]

where $\sigma_g \in S_m$ is a permutation on the roots of $g$ induced by $\text{Frob}_q$. Taking $\chi$ to be the character of $\text{Ind}_{S_{\tau}}^{S_m}(1)$, we obtain:
4.1. Proposition.

\[ |\text{Poly}_n(\lambda)/\mathbb{F}_q| = \sum_{g \in \text{Poly}_m/\mathbb{F}_q} \chi(g) \]

Proposition 4.1 is Corollary 1.5.1 in disguise! For a square-free polynomial \( g \in \text{Poly}_m/\mathbb{F}_q \), write \( g = g_1g_2 \cdots g_s \) for its factorization into irreducibles. The partition \( \mu =: (\mu_1, \mu_2, \ldots, \mu_s) \vdash m \) given by \( \mu_i = \deg g_i \) is the cycle type of \( \sigma_g \), and \( \chi(\sigma_g) \) is the number of Young tableaux of type \( \tau \) fixed by \( \sigma_g \). Let \( \sigma_g = \sigma_1\sigma_2 \cdots \sigma_s \) be a cycle decomposition in which \( \sigma_i \) has order \( \mu_i \). Each Young tableau of type \( \tau \) is fixed by \( \sigma_g \) if and only if each row of the tableau is a union of entries of \( \sigma_i \)'s. The number of such Young tableaux is the number of ‘ways of refining’ the partition \( \tau \) to the partition \( \mu \), i.e. the number of lists \( (\Sigma_1, \Sigma_2, \ldots, \Sigma_r) \) of \( r \) disjoint subsets partitioning the set \( \{1, 2, \ldots, s\} \) such that

\[ (\mu_k)_{k \in \Sigma_j} \]

is a partition of \( \tau_j \) for \( j = 1, 2, \ldots, r \). Each \( \Sigma_j \) determines a factor \( f_j \) of \( g \) of degree \( \tau_j \) given by

\[ f_j = \prod_{k \in \Sigma_j} g_k \]

and vice versa. So \( \chi(\sigma_g) \) is the number of lists \( (f_1, f_2, \ldots, f_r) \) of factors of \( g \) with \( \deg f_j = \tau_j \). But this is exactly the size of the fiber over \( g \) of the multiplication map \( \text{Pol}_r \to \text{Pol}_m \)

\[ (f_j) \mapsto \prod f_j \]

thus recovering Corollary 1.5.1.

Remark. Using equation (4.3), we may rewrite the formula (4.2) as

\[ |\text{Poly}_n(\lambda)/\mathbb{F}_q| = q^m \sum_{i \geq 0} (-q)^i \left( H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C}), 1 \right)_{S_r} \]

\[ = q^m \sum_{\gamma \in S_r} \sum_{i \geq 0} (-q)^i \text{Trace}(\gamma, H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})) \]

Lehrer studied the polynomial

\[ \sum_{i \geq 0} t^i \text{Trace}(\gamma, H^i(\text{Conf}_m/\mathbb{C}; \mathbb{C})) \]

in \( t \), called the Poincaré series of \( \gamma \), and gave an explicit formula (Theorem 5.5 in [8]), which, evaluated at \( t = -q \), gives another count for \( |\text{Poly}_n(\lambda)/\mathbb{F}_q| \).

References

[1] N. Sahasrabudhe, Grothendieck Ring of Varieties