EILENBERG-MACLANE SPACES AS A LINK BETWEEN COHOMOLOGY AND HOMOTOPY

TAYLOR SUTTON

Abstract. This paper gives an exposition of an established theorem in algebraic topology: there exists an isomorphism between reduced cohomology groups of a given CW complex and basepoint-preserving homotopy classes of maps from that CW complex to a suitable Eilenberg-MacLane space. In particular, we show an isomorphism $H^n(X; G) \cong \langle X, K(G, n) \rangle$.

Contents

1. Introduction 1
2. The Axioms for Cohomology 1
3. Verifying the Axioms 2
4. Wrapping up 3
Acknowledgments 3
References 3

1. Introduction

The relationship between cohomology and homotopy—both fundamental concepts—is a natural object of interest in algebraic topology. Eilenberg-MacLane spaces provide one fruitful description of this relationship. An Eilenberg-MacLane space $K(G, n)$ has the property that $\pi_n(K(G, n)) \cong G$ and every other homotopy group is trivial. Verifying the axioms for cohomology for $\langle X, K(G, n) \rangle$, the set of basepoint-preserving homotopy classes of maps from a CW complex $X$ to the Eilenberg-MacLane space $K(G, n)$, allows, with a little extra work, the demonstration of an isomorphism $H^n(X; G) \cong \langle X, K(G, n) \rangle$. The more precise statement of the theorem is as follows.

Theorem 1.1. For all CW complexes $X$ and $n > 0$, there are natural bijections $T : \langle X, K(G, n) \rangle \to H^n(X; G)$ with $G$ any abelian group. If $H^n$ is a reduced cohomology theory, the restriction on $n$ may be omitted.

The approach to the above demonstration will largely follow the one given in Hatcher’s Algebraic Topology with adjustments and rearrangements made when necessary to clarify the central theorem as a standalone topic rather than a subordinate element in a larger text. The succeeding section will lay out the axioms for cohomology, which will be followed by a section checking these axioms for $\langle X, K(G, n) \rangle$. The final section deals with the coefficient group and wraps up the proof of the theorem.

Date: August 28, 2015.
2. The Axioms for Cohomology

A reduced cohomology theory of CW complexes \((X, A)\) is a contravariant functor \(\tilde{h}^n\) to abelian groups with natural coboundary homomorphisms \(\delta : \tilde{h}^n(X) \to \tilde{h}^{n+1}(X/A)\) satisfying three axioms. The axioms uniquely determine the cohomology theory.

**Homotopy Axiom:** Given \(f, g : X \to Y\), if \(f \simeq g\), then \(f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)\).

**Wedge Sum Axiom:** For a wedge sum \(X = \bigsqcup X_\alpha\) with inclusions \(i_\alpha : X_\alpha \hookrightarrow X\), the map \(\prod \alpha i_\alpha : \tilde{h}^n(X) \to \prod \alpha \tilde{h}^n(X_\alpha)\) is an isomorphism for each \(n\).

**Long Exact Sequence Axiom:** For each CW pair \((X, A)\) there exists a long exact sequence
\[
\cdots \delta \to \tilde{h}^n(X/A) \xrightarrow{q} \tilde{h}^n(X) \xrightarrow{i} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q} \cdots
\]

3. Verifying the Axioms

The first task is to verify that \(\langle X, K(G, n) \rangle\) satisfies the three axioms for cohomology outlined in Section 2. To that end, a couple of definitions are useful.

**Definition 3.1.** An \(\Omega\)-spectrum is a sequence of CW complexes \(\{K_n\}\) with weak homotopy equivalences \(K_n \to \Omega K_{n+1}\) for all \(n\).

**Definition 3.2.** A cofibration sequence of a CW pair \((X, A)\) is a sequence of the following form.
\[
A \hookrightarrow X \to X/A \hookrightarrow \Sigma A \hookrightarrow \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \hookrightarrow \Sigma^2 X \to ...
\]

Armed with these definitions, we proceed to the main object of this section.

**Theorem 3.3.** Given an \(\Omega\)-spectrum \(\{K_n\}\), the functors \(X \mapsto \tilde{h}^n(X) = \langle X, K_n \rangle\) for integer \(n\) define a reduced cohomology theory on the category of pointed CW complexes and basepoint-preserving maps.

**Proof.** First consider the homotopy axiom. A basepoint-preserving map \(f : X \to Y\) induces a map \(f^* : \langle Y, K_n \rangle \to \langle X, K_n \rangle\) which sends a map \(Y \to K_n\) to a map \(X \xrightarrow{f} Y \to K_n\). This \(f^*\) depends only on the homotopy class of \(f\); moreover, regarding \(K_n\) as \(\Omega K_{n+1}\) and using the concatenation of loops as a group operation makes it apparent that \(f^*\) is a homomorphism. This confirms the homotopy axiom.

Verification of the wedge sum axiom is equally brief. The axiom holds since a map \(\bigsqcup X_\alpha \to K_n\) is just a collection of basepoint-preserving maps \(X_\alpha \to K_n\).

Moving to the final axiom, let \((X, A)\) be a CW pair and \(CZ\) denote the cone on a CW complex \(Z\). Then the successive addition of mapping cones produces the following sequence of inclusions.
\[
A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA)
\]

Making use of homotopy equivalences yields a more instructive sequence—the cofibration sequence defined above.
\[
A \hookrightarrow X \hookrightarrow X/A \hookrightarrow \Sigma A \hookrightarrow \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \hookrightarrow \Sigma^2 X \to ...
\]
Fixing a space \( K \) and defining maps by composition with this sequence gives a new sequence of basepoint-preserving homotopy classes of maps.

\[
\langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X/A, K \rangle \leftarrow \langle \Sigma A, K \rangle \leftarrow \langle \Sigma X, K \rangle \leftarrow \ldots
\]

This sequence is exact. Since the method of successively adding cones determines a term by its two predecessors, a proof of the exactness of the whole sequence reduces to a proof of the exactness of \( \langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X \cup CA, K \rangle \). In this case, consider a map \( X \rightarrow K \). That this map goes to zero in \( \langle A, K \rangle \) means that its restriction to \( A \) is nullhomotopic, which is the same as its extending to a map \( X \cup CA \rightarrow K \). Thus, exactness is established. Therefore, if \( K = K(G, n) \) for some \( G \) and \( n \), then applying the relation \( \langle \Sigma X, K \rangle = \langle X, \Omega K \rangle \) to the entire sequence gives the desired long exact sequence. \( \square \)

4. Wrapping up

With the results of Section 3, the proof of Theorem 1.1 can be completed. This section follows the discussion in Frankland [2, p.8].

Proof. (Of Theorem 1.1) By Theorem 3.3 and the uniqueness of cohomology theories under the axioms of Section 2, for arbitrary CW complex \( X \) and for abelian group \( G \), we have the following.

\[
\langle X, K(G, n) \rangle \cong H^n(X; \langle S^0, K(G, 0) \rangle)
\]

By the definition of an Eilenberg-MacLane space we know the following.

\[
\langle S^0, K(G, 0) \rangle = \pi_0 K(G, 0) \cong G
\]

Hence, \( \langle X, K(G, n) \rangle \cong H^n(X; G) \), which is the assertion of Theorem 1.1. \( \square \)

Acknowledgments. I owe a great deal of thanks to Peter May, Inna Zakharevich, and Agnes Beaudry, who served as my first (and able) instructors in algebraic topology. Each a fine teacher in his or her own right, it is no small thing that, combined, they were greater than the sum of their parts. They kept a novice engaged and helped him along. Special appreciation is due my mentor, Foling Zou, who was an exceedingly valuable resource. Initiating someone in the details of topic of which he was formerly ignorant takes a great deal of effort and patience. I do not delude myself that I am a master of algebraic topology. Nevertheless, there is a fairly great leap to be made between taking a single undergraduate math class and writing this paper, and no one is more responsible for my progress in this respect than her.

References