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Chapter 1

Representations

1.1 Basic Definitions

Groups are intended to describe symmetries of geometric and other mathematical objects. Representations are symmetries of some of the most basic objects in geometry and algebra, namely vector spaces.

Representations have three different aspects — geometric, numerical and algebraic — and manifest themselves in corresponding form. We begin with the numerical form.

In a general context we write groups $G$ in multiplicative form. The group structure (multiplication) is then a map $G \times G \to G, (g, h) \mapsto g \cdot h = gh$, the unit element is $e$ or $1$, and $g^{-1}$ is the inverse of $g$.

An $n$-dimensional matrix representation of the group $G$ over the field $K$ is a homomorphism $\varphi : G \to GL_n(K)$ into the general linear group $GL_n(K)$ of invertible $(n, n)$-matrices with entries in $K$. Two such representations $\varphi, \psi$ are said to be conjugate if there exists a matrix $A \in GL_n(K)$ such that the relation $A \varphi(g) A^{-1} = \psi(g)$ holds for all $g \in G$. The representation $\varphi$ is called faithful if $\varphi$ is injective. If $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}$, we talk about complex, real, and rational representations.

The group $GL_1(K)$ will be identified with the multiplicative group of non-zero field elements $K^* = K \setminus \{0\}$. In this case we are just considering homomorphisms $G \to K^*$.

Next we come to the geometric form of a representation as a symmetry group of a vector space. The field $K$ will be fixed.

A representation of $G$ on the $K$-vector space $V$, a $KG$-representation for short, is a map

$$\rho : G \times V \to V, \quad (g, v) \mapsto \rho(g, v) = g \cdot v = gv$$

with the properties:
(1) \( g(hv) = (gh)v, ev = v \) for all \( g, h \in G \) and \( v \in V \).

(2) The left translation \( l_g : V \to V, v \mapsto gv \) is a \( K \)-linear map for each \( g \in G \).

We call \( V \) the representation space. Its dimension as a vector space is the dimension \( \dim V \) of the representation (sometimes called the degree of the representation). The rules (1) are equivalent to \( l_g \circ l_h = l_{gh} \) and \( l_e = \text{id}_V \).

They express the fact that \( \rho \) is a group action – see the next section. From \( l_g l_g^{-1} = l_{gg^{-1}} = l_e = \text{id} \) we see that \( l_g \) is a linear isomorphism with inverse \( l_g^{-1} \).

Occasionally it will be convenient to define a representation as a map \( V \times G \to V, (v,g) \mapsto vg \) with the properties \( v(hg) = (vh)g \) and \( ve = v \), and \( K \)-linear right translations \( r_g : v \mapsto vg \). These will be called right representations as opposed to left representations defined above. The map \( r_g : v \mapsto vg \) is then the right translation by \( g \). Note that now \( r_g \circ r_h = r_{hg} \) (contravariance). If \( V \) is a right representation, then \( (g,v) \mapsto vg^{-1} \) defines a left representation. We work with left representations if nothing else is specified.

One can also use both notions simultaneously. A \((G,H)\)-representation is a vector space \( V \) with the structure of a left \( G \)-representation and a right \( H \)-representation, and these structures are assumed to commute \( (gv)h = g(vh) \).

A morphism \( f : V \to W \) between \( KG \)-representations is a \( K \)-linear map \( f \) which is \( G \)-equivariant, i.e., which satisfies \( f(gv) = gf(v) \) for \( g \in G \) and \( v \in V \). Morphisms are also called intertwining operators. A bijective morphism is an isomorphism. The vector space of all morphisms \( V \to W \) is denoted \( \text{Hom}_G(V,W) = \text{Hom}_{KG}(V,W) \). Finite-dimensional \( KG \)-representations and their morphisms form a category \( KG\text{-Rep} \).

Let \( V \) be an \( n \)-dimensional representation of \( G \) over \( K \). Let \( B \) be a basis of \( V \) and denote by \( \varphi^B(g) \in GL_n(K) \) the matrix of \( l_g \) with respect to \( B \). Then \( g \mapsto \varphi^B(g) \) is a matrix representation of \( G \). Conversely, from a matrix representation we get in this manner a representation.

\[ (1.1.1) \text{ Proposition.} \quad \text{Let } V,W \text{ be representations of } G, \text{ and } B,C \text{ bases of } V,W. \text{ Then } V,W \text{ are isomorphic if and only if the corresponding matrix representations } \varphi^B, \varphi^C \text{ are conjugate.} \]

\[ \text{Proof. Let } f : V \to W \text{ be an isomorphism and } A \text{ its matrix with respect to } B,C. \text{ The equivariance } f \circ l_g = l_g \circ f \text{ then translates into } A\varphi^B(g) = \varphi^C(g)A; \text{ and conversely.} \]

Conjugate 1-dimensional representations are equal. Therefore the isomorphism classes of 1-dimensional representations correspond bijectively to homomorphisms \( G \to K^* \). The aim of representation theory is not to determine
matrix representations. But certain concepts are easier to explain with the help of matrices.

Let $V$ be a representation of $G$. A sub-representation of $V$ is a subspace $U$ which is $G$-invariant, i.e., $gu \in U$ for $g \in G$ and $u \in U$. A non-zero representation $V$ is called irreducible if it has no sub-representations other than $\{0\}$ and $V$. A representation which is not irreducible is called reducible.

(1.1.2) **Schur’s Lemma.** Let $V$ and $W$ be irreducible representations of $G$.

1. A morphism $f: V \rightarrow W$ is either zero or an isomorphism.
2. If $K$ is algebraically closed then a morphism $f: V \rightarrow V$ is a scalar multiple of the identity, $f = \lambda \cdot \text{id}$.

**Proof.** (1) Kernel and image of $f$ are sub-representations. If $f \neq 0$, then the kernel is different from $V$ hence equal to $\{0\}$ and the image is different from $\{0\}$ hence equal to $W$.

(2) Algebraically closed means: Non-constant polynomials have a root. Therefore $f$ has an eigenvalue $\lambda \in K$ (root of the characteristic polynomial). Let $V(\lambda)$ be the eigenspace and $v \in V(\lambda)$. Then $f(\lambda v) = \lambda f(v) = \lambda(\lambda v) = \lambda^2 v$. Therefore $\lambda v \in V(\lambda)$, and $V(\lambda)$ is a sub-representation. By irreducibility, $V = V(\lambda)$.

(1.1.3) **Proposition.** An irreducible representation of an abelian group $G$ over an algebraically closed field is one-dimensional.

**Proof.** Since $G$ is abelian, the $l_g$ are morphisms and, by 1.1.2, multiples of the identity. Hence each subspace is a sub-representation.

(1.1.4) **Example.** Let $S_n$ be the symmetric group of permutations of $\{1, \ldots, n\}$. We obtain a right(!) representation of $S_n$ on $K^n$ by permutation of coordinates

$$K^n \times S_n \rightarrow K^n, \quad ((x_1, \ldots, x_n), \sigma) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This representation is not irreducible if $n > 1$. It has the sub-representations $T_n = \{(x_i) \mid \sum_{i=1}^n x_i = 0\}$ and $D = \{(x, \ldots, x) \mid x \in K\}$.

Schur’s lemma can be expressed in a different way. Recall that an algebra $A$ over $K$ consists of a $K$-vector space together with a $K$-bilinear map $A \times A \rightarrow A, (a, b) \mapsto ab$ (the multiplication of the algebra). The algebra is called associative (commutative), if the multiplication is associative (commutative). An associative algebra with unit element is therefore a ring with the additional property that the multiplication is bilinear with respect to the scalar multiplication in the vector space. In a division algebra (also called skew field) any non-zero element has a multiplicative inverse. A typical example of
an associative algebra is the endomorphism algebra \( \text{Hom}_G(V,V) \) of a representation \( V \); multiplication is the composition of endomorphisms. Other examples are the algebra \( M_n(K) \) of \((n,n)\)-matrices with entries in \( K \) and the polynomial algebra \( K[x] \). The next proposition is a reformulation of Schur’s lemma.

\begin{proposition}
Let \( V \) be an irreducible \( G \)-representation. Then the endomorphism algebra \( A = \text{Hom}_G(V,V) \) is a division algebra. If \( K \) is algebraically closed, then \( K \to A, \lambda \mapsto \lambda \cdot \text{id} \) is an isomorphism of \( K \)-algebras. \end{proposition}

2. Let \( V \) be an irreducible \( G \)-representation over \( \mathbb{R} \). A finite-dimensional division algebra over \( \mathbb{R} \) is one of the algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H} \). We call \( V \) of \text{real, complex, quaternionic type} according to the type of its endomorphism algebra.

The third form of a representation — namely a module over the group algebra — will be introduced later.

\begin{proposition}
Let \( V \) be a \text{cyclic group}. The cyclic group of order \( n \) is the additive group \( \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n \) of integers modulo \( n \). We also use a formal multiplicative notation for this group \( C_n = \langle a \mid a^n = 1 \rangle \); this means: \( a \) is a generator and the \( n \)-th power is the unit element.

Homomorphisms \( \alpha : C_n \to H \) into another group \( H \) correspond bijectively to elements \( h \in H \) such that \( h^n = 1 \), via \( a \mapsto \alpha(a) \). Hence there are \( n \) different \( 1 \)-dimensional representations over the complex numbers \( \mathbb{C} \), given by \( a \mapsto \exp(2\pi it/n) \), \( 0 \leq t < n \).

The rotation matrices \( D(\alpha) \)
\[
D(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
satisfy \( D(\alpha)D(\beta) = D(\alpha + \beta) \) and \( BD(\alpha)B^{-1} = D(-\alpha) \). We obtain a 2-dimensional real representation \( \varphi_1 : a \mapsto D(2\pi t/n) \). The representations \( \varphi_t \) and \( \varphi_{-t} = \varphi_{n-t} \) are conjugate.

\begin{proposition}
Let \( V \) be a \text{dihedral group}. Groups can be presented in terms of generators and relations. We do not enter the theory of such presentations but consider an example. Let \( D_{2n} = \langle a, b \mid a^n = b^2, bab^{-1} = a^{-1} \rangle \).

This means: The group is generated by two elements \( a \) and \( b \), and these generators satisfy the specified relations. The \text{universal property} of this presentation is: The homomorphisms \( \alpha : D_{2n} \to H \) into any other group \( H \) correspond bijectively to pairs \( (A = \alpha(a), B = \alpha(b)) \) in \( H \) such that \( A^n = 1 = B^2, BAB^{-1} = A^{-1} \).

Thus \( 1 \)-dimensional representations over \( \mathbb{C} \) correspond to complex numbers \( A, B \) such that \( A^n = B^2 = A^2 = 1 \). If \( n \) is odd there are two pairs \( (1, \pm 1) \); if \( n \) is even there are four pairs \( (\pm 1, \pm 1) \).
A 2-dimensional representation on the \( \mathbb{R} \)-vector space \( \mathbb{C} \) is specified by \( a \cdot z = \lambda z, \; b \cdot z = \bar{z} \) where \( \lambda^n = 1 \). Complex conjugation shows that the representations which correspond to \( \lambda \) and \( \bar{\lambda} \) are isomorphic. Denote the representation obtained from \( \lambda = \exp(2\pi it/n) \) by \( V_t \).

The group \( D_{2n} \) has order \( 2n \) and is called the **dihedral group** of this order. From a geometric viewpoint, \( D_{2n} \) is the orthogonal symmetry group of the regular \( n \)-gon in the plane. A faithful matrix representation \( D_{2n} \to O(2) \) is obtained by choosing \( \lambda = \exp(2\pi i/n) \). The powers of \( a \) correspond to rotations, the elements \( a^t b \) to reflections.

**Example.** The real representations \( \varphi_t, 1 \leq t < n/2 \), of \( C_n \) in 1.1.6 are irreducible. A nontrivial sub-representation would be one-dimensional and spanned by an eigenvector of \( \varphi_t(a) \).

If we consider \( \varphi_t \) as a complex representation, then it is no longer irreducible, since eigenvectors exist. In terms of matrices

\[
PD(\alpha)P^{-1} = \begin{pmatrix} \exp(i\alpha) & 0 \\ 0 & \exp(-i\alpha) \end{pmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\]

The representations \( V_t, 1 \leq t < n/2 \) of \( D_{2n} \) in 1.1.7 are irreducible, since they are already irreducible as representations of \( C_n \). But this time they remain irreducible when considered as complex representations. The reason is, that \( PB^tP^{-1} \) does not preserve the eigenspaces.

**Problems**

1. The dihedral group \( D_{2n} \) has the presentation \( \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle \).
2. Recall the notion of a semi-direct product of groups and show that \( D_{2n} \) is the semi-direct product of \( C_n \) by \( C_2 \).
3. Let \( Q_{4n} = \langle a, b \mid a^n = b^2, bab^{-1} = a^{-1} \rangle, \; n \geq 2 \). Deduce from the relation \( b^4 = a^{2n} = 1 \). Show that \( Q_{4n} \) is a group of order \( 4n \). Show that \( a \mapsto \exp(\pi i/n), b \mapsto j \) induces an isomorphism of \( Q_{4n} \) with a subgroup of the multiplicative group of the quaternions. The group \( Q_{4n} \) is called a **quaternion group**. Show that \( Q_{4n} \) has the presentation \( \langle s, t \mid s^2 = t^2 = (st)^n \rangle \).
   Construct a two-dimensional faithful irreducible (matrix) representation over \( \mathbb{C} \).
4. Let \( A_4 \) be the alternating group of order 12 (even permutations in \( S_4 \)). Show: \( A_4 \) has 3 elements of order 2, 8 elements of order 3. Show that \( A_4 \) is the semi-direct product of \( C_2 \times C_2 \) by \( C_3 \). Show \( A_4 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle \); show that \( s = ab \) and \( t = ba \) are commuting elements of order 2 which generate a normal subgroup. Show that the matrices

\[
a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.
\]
Representations

A representation of $C_2$ on $V$ amounts to specifying an involution $T: V \to V$, i.e., a linear map $T$ with $T^2 = \text{id}$. If the field $K$ has characteristic different from 2 show that $V$ is the direct sum of the $\pm 1$-eigenspaces of $T$. (Consider the operators $\frac{1}{2}(1+T)$ and $\frac{1}{2}(1-T)$.)

1.2 Group Actions and Permutation Representations

In this section we collect basic terminology about group actions. We use group actions to construct the important class of permutation representations.

Let $G$ be a multiplicative group with unit element $e$. A left action of a group $G$ on a set $X$ is a map $\rho: G \times X \to X$, $(g,x) \mapsto \rho(g,x) = gx$ with the properties $g(hx) = (gh)x$ and $ex = x$ for $g,h \in G$ and $x \in X$. The pair $(X,\rho)$ is called a (left) $G$-set. Each $g \in G$ yields the left translation $l_g: X \to X, x \mapsto gx$ by $g$. It is a bijection with inverse the left translation by $g^{-1}$. An action is called effective, if $l_g$ for $g \neq e$ is never the identity. We also use (right) actions $X \times G \to X, (x,g) \mapsto xg$. They satisfy $(xh)g = x(hg)$ and $xe = x$. Usually we work with left $G$-actions.

A subset $A$ of a $G$-set $X$ is called $G$-stable or $G$-invariant, if $g \in G$ and $a \in A$ implies $ga \in A$.

Recall that we defined a representation as a group action on a vector space with the additional property that the left translations are linear maps. We now use group actions to construct representations.

Let $S$ be a finite (left) $G$-set and denote by $KS$ the vector space with $K$-basis $S$. Thus elements in $KS$ are linear combinations $\sum_{s \in S} \lambda_s s$ with $\lambda_s \in K$. The left action of $G$ on $S$ is extended linearly to $KS$

$$g \cdot \left( \sum_{s \in S} \lambda_s s \right) = \sum_{s \in S} \lambda_s (g \cdot s) = \sum_{s \in S} \lambda_{g^{-1}x} s.$$

The resulting representation is called the permutation representation of $S$. An important example is obtained from the group $G = S$ with left action by group multiplication. The associated permutation representation is the left regular representation of the finite group $G$. Right multiplication leads to the right regular representation.

Let $X$ be a $G$-set. Then $R = \{(x,gx) \mid x \in X, g \in G\}$ is an equivalence relation on $X$. Let $X/G$ denote the set of equivalence classes. The class of $x$ is $Gx = \{gx \mid g \in G\}$ and called the orbit through $x$. We call $X/G$ the orbit set or (orbit space) of the $G$-set $X$. An action is called transitive, if
it consists of a single orbit. For systematic reasons it would be better to denote the orbit set of a left action by $G \backslash X$. If right and left actions occur, we use both notations.

A group acts on itself by conjugation $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$. Elements are \textbf{conjugate} if they are in the same orbit. The orbits are called \textbf{conjugation classes}. A function on $G$ is called a \textbf{class function}, if it is constant on conjugacy classes.

Let $H$ be a subgroup of $G$. We have the set $G/H$ of right cosets $gH$ with left $G$-action by left translation

$$G \times G/H \to G/H, \quad (k, gH) \mapsto kgH.$$ 

$G$-sets of this form are called \textbf{homogeneous} $G$-sets. Similarly, we have the set $H \backslash G$ of left cosets $Hg$ with an action by right translation.

We write $H \leq G$, if $H$ is a subgroup of $G$, and $H < G$, if it is a proper subgroup. On the set $\text{Sub}(G)$ of subgroups of $G$ the relation $\leq$ is a partial order.

The group $G$ acts on $\text{Sub}(G)$ by conjugation $(g, H) \mapsto gHg^{-1} = gH$. The orbit through $H$ consists of the subgroups $H$ of $G$ which are \textbf{conjugate} to $H$. We write $K \sim L$ or $K \sim_{G} L$, if there exists $g \in G$ such that $gKg^{-1} = L$. We denote by $(H)$ the conjugacy class of $H$. Let $\text{Con}(G)$ be the set of conjugacy classes of subgroups of $G$. We say $H$ is \textbf{subconjugate} to $K$ in $G$, if $H$ is conjugate in $G$ to a subgroup of $K$. We denote this fact by $(H) \leq (K)$; and by $(H) < (K)$, if equality is excluded.

The \textbf{stabilizer} or \textbf{isotropy group} of $x \in X$ is the subgroup $G_x = \{ g \in G \mid gx = x \}$. We have $G_{gx} = gG_xg^{-1}$. An action is called \textbf{free}, if all isotropy groups are trivial. The set of isotropy groups of $X$ is denoted $\text{Iso}(X)$.

A \textbf{family $\mathcal{F}$ of subgroups} is a subset of $\text{Sub}(G)$ which consists of complete conjugacy classes. If $\mathcal{F}$ and $\mathcal{G}$ are families, we write $\mathcal{F} \circ \mathcal{G}$ for the family of intersections $\{ K \cap L \mid K \in \mathcal{F}, L \in \mathcal{G} \}$. We call $\mathcal{F}$ \textbf{multiplicative}, if $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$, and $\mathcal{G}$ is called $\mathcal{F}$-\textbf{modular}, if $\mathcal{F} \circ \mathcal{G} \subset \mathcal{G}$. A family is called \textbf{closed}, if it contains with a group all supergroups, and it is called \textbf{open}, if it contains with a group all subgroups. Let $(\mathcal{F})$ denote the set of conjugacy classes of $\mathcal{F}$. Suppose $\text{Iso}(X) \subset \mathcal{F}$, then we call $X$ an $\mathcal{F}$-\textbf{set}. We denote by $X(\mathcal{F})$ the subset of points in $X$ with isotropy groups in $\mathcal{F}$.

A $G$-\textbf{map} $f : X \to Y$ between $G$-sets, also called a $G$-\textbf{equivariant map}, is a map which satisfies $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Left $G$-sets and $G$-equivariant maps form the category $G$-\textbf{SET}. By passage to orbits, a $G$-map $f : X \to Y$ induces $f/G : X/G \to Y/G$. The category $G$-\textbf{SET} of $G$-sets and $G$-maps has products: If $(X_j \mid j \in J)$ is a family of $G$-sets, then the Cartesian product $\prod_{j \in J} X_j$ with so-called \textbf{diagonal action} $g(x_j) = (gx_j)$ is a product in this category.
(1.2.1) Proposition. Let $C$ be a transitive $G$-set and $c \in C$. Then $G/G_c \rightarrow C, gc \mapsto gc$ is a well-defined isomorphism of $G$-sets (a simple algebraic verification). The orbits of a $G$-set are transitive. Therefore each $G$-set is isomorphic to a disjoint sum of homogeneous $G$-sets.

For a $G$-set $X$ and a subgroup $H$ of $G$ we use the following notations

$$X_H = \{ x \in X \mid G_x = H \},$$
$$X_{(H)} = \{ x \in X \mid (G_x) = (H) \},$$
$$X^H = \{ x \in X \mid hx = x, h \in H \},$$
$$X^{>H} = X^H \setminus X_H,$$
$$X^{(H)} = G X^H = \{ x \in X \mid (H) \leq (G_x) \},$$
$$X^{>(H)} = X^{(H)} \setminus X_{(H)}.$$

We call $X^H$ the $H$-fixed point set of $X$. If $f: X \rightarrow Y$ is a $G$-map, then $f(X^H) \subset Y^H$. The left translation $l_g: X \rightarrow X$ induces a bijection $X^H \rightarrow X^K, K = gHg^{-1}$. The subset $X_{(H)}$ is $G$-stable; it is called the $(H)$-orbit bundle of $X$.

(1.2.2) Example. The permutation representation $KS$ is always reducible ($|S| > 1$). The fixed point set is a non-zero proper sub-representation. The dimension of $(KS)^G$ is $|S/G|$; a basis of $(KS)^G$ consists of the $x_C = \sum_{s \in C} s$ where $C$ runs through the orbits of $S$.

Suppose $X$ is a right and $Y$ a left $H$-set. Then $X \times_H Y$ denotes the quotient of $X \times Y$ with respect to the equivalence relation $(xh, y) \sim (x, hy), h \in H$. This is the orbit set of the action $(h, (x, y)) \mapsto (xh^{-1}, hy)$ of $H$ on $X \times Y$.

Let $G$ and $H$ be groups. A $(G, H)$-set $X$ is a set $X$ together with a left $G$-action and a right $H$-action which commute $(gx)h = g(xh)$. If we form $X \times_H Y$, then this set carries an induced $G$-action $g \cdot (x, y) = (gx, y)$. If $f: Y_1 \rightarrow Y_2$ is an $H$-map, then we obtain an induced $G$-map $X \times_H f: X \times_H Y_1 \rightarrow X \times_H Y_2$. This construction yields a functor $\rho(X): H\text{-SET} \rightarrow G\text{-SET}$.

We apply this construction to the $(G, H)$-set $G = X$ with action by left $G$-translation and right $H$-translation for $H \leq G$. The resulting functor is called induction functor

$$\text{ind}^G_H: H\text{-SET} \rightarrow G\text{-SET}.$$ 

It is left adjoint to the restriction functor

$$\text{res}^G_H: G\text{-SET} \rightarrow H\text{-SET},$$

given by considering a $G$-set as an $H$-set. The adjointness means that there is a natural bijection

$$\text{Hom}_G(\text{ind}^G_H X, Y) \cong \text{Hom}_H(X, \text{res}^G_H Y).$$
It assigns to an $H$-map $f: X \to Y$ the $G$-map $G \times_H X \to Y$, $(g, x) \mapsto gf(x)$.

One can obtain interesting group theoretic results by counting orbits and fixed points. We give some examples.

(1.2.3) Proposition. Let $P$ be a $p$-group and $X$ a finite $P$-set; then $|X| \equiv |X^P| \mod p$. Let $C$ be cyclic of order $p^t$ and $D \leq C$ the unique subgroup of order $p$; then $|X| \equiv |X^D| \mod p^t$.

Proof. Each orbit in $X \setminus X^P$ has cardinality divisible by $p$. Each orbit in $X \setminus X^D$ has cardinality $p^t$.

(1.2.4) Proposition. Let $P \neq 1$ be a $p$-group. Then $P$ has a non-trivial center $Z(P) = \{x \in P \mid \forall y \in P, xy = yx\}$.

Proof. Let $P$ act on itself by conjugation $(x, y) \mapsto xyx^{-1}$. The fixed point set is the center. Since $1 \in Z(P)$, we see from 1.2.3 that $|Z(P)|$ is non-zero and divisible by $p$.

(1.2.5) Proposition. Let $P$ be a $p$-group. There exists a chain of normal subgroups $P_i \triangleleft P$

$1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_r = P$

such that $|P_i/P_{i-1}| = p$.

Proof. Induct on $|P|$. Since subgroups of the center are normal, there exists by 1.2.4 a normal subgroup $P_1$ of order $p$. Apply the induction hypothesis to the factor group $P/P_1$ and lift a normal series to $P$.

(1.2.6) Proposition. Let $K$ be a field of characteristic $p$ and $V$ a $KP$-representation for a $p$-group $P$. Then $V^P \neq \{0\}$.

Proof. Let $P$ have order $p$ with generator $x$. Then $l_x: V \to V$ has eigenvalues a root of $X^p - 1 = (X - 1)^p$. Hence $1$ is the only eigenvalue. In the general case choose a normal subgroup $Q \triangleleft P$ and observe that $V^Q$ is a $K(P/Q)$-representation.

A group $A$ is called **elementary abelian of rank** $n$ if it is isomorphic to the $n$-fold product $(C_p)^n$ of cyclic groups $C_p$ of prime order $p$. We can view this group as $n$-dimensional vector space over the prime field $\mathbb{F}_p$.

(1.2.7) Proposition. Let the $p$-group $P$ act on the elementary abelian $p$-group $A$ of rank $n$ by automorphisms. Then there exists a chain $1 = A_0 \triangleleft \ldots \triangleleft A_n = A$ of subgroups which are $P$-invariant and $|A_i/A_{i-1}| = p$.

(1.2.8) Counting lemma. Let $G$ be a finite group, $M$ a finite $G$-set, and $\langle g \rangle$ the cyclic subgroup generated by $g \in G$. Then $|G| \cdot |M/G| = \sum_{g \in G} |M^{(g)}|$.
Proof. Let \( X = \{(g, x) \in G \times M \mid gx = x\} \). Consider the maps

\[
p: X \to G, \quad (g, x) \mapsto g, \quad q: X \to M/G, \quad (g, x) \mapsto Gx.
\]

Since \( p^{-1}(g) = |\{g\} \times M| \), the right hand side is the sum of the cardinalities of the fibres of \( p \). Since \( |q^{-1}(Gx)| = |Gx||Gx| = |G| \), the left hand side is the sum of the cardinalities of the fibres of \( q \).

Problems

1. Let \( X \) and \( Y \) be \( G \)-sets. The set \( \text{Hom}(X, Y) \) of all maps \( X \to Y \) carries a left \( G \)-action \( (g \cdot f)(x) = gf(g^{-1}x) \). The \( G \)-fixed point set \( \text{Hom}(X, Y)^G \) is the subset of \( G \)-maps \( X \to Y \).

2. Let \( X \) be a \( G \)-set and \( H \leq G \). Then \( G \times_H X \to G/H \times X, \quad (g, x) \mapsto (gH, gx) \) is a bijection of \( G \)-sets. If \( Y \) is a further \( H \)-set, then we have an isomorphism of \( G \)-sets \( G \times_H (X \times Y) \cong X \times (G \times_H Y) \).

3. Determine the conjugacy classes of \( D_{2n} \) and \( Q_{4n} \).

4. The orbits of \( G/K \times G/L \) correspond bijectively to the double cosets \( K \backslash G/L \); the maps

\[
G \backslash (G/K \times G/L) \to K \backslash G/L, \quad G \cdot (uK, vL) \mapsto Ku^{-1}vL
\]

\[
K \backslash G/L \to G \backslash (G/K \times G/L), \quad v \mapsto G \cdot (eK, vL)
\]

are inverse bijections.

1.3 The Orbit Category

The full subcategory of \( G \)-SET with object the homogeneous \( G \)-sets is called the orbit category \( \text{Or}(G) \) of \( G \).

(1.3.1) Proposition. Let \( H \) and \( K \) be subgroups of \( G \).

1. There exists a \( G \)-map \( G/H \to G/K \) if and only if \( (H) \) is subconjugate to \( (K) \).

2. Each \( G \)-map \( G/H \to G/K \) has the form \( R_a: gH \mapsto gaK \) for an \( a \in G \) such that \( a^{-1}Ha \subset K \).

3. \( R_a = R_b \) if and only if \( a^{-1}b \in K \).

4. \( G/H \) and \( G/K \) are \( G \)-isomorphic if and only if \( H \) and \( K \) are conjugate in \( G \).

Proof. Let \( f: G/H \to G/K \) be equivariant and suppose \( f(eH) = aK \). By equivariance, we have for all \( h \in H \) the equalities \( aK = f(eH) = f(hH) = hf(eH) = haK \) and hence \( a^{-1}Ha \subset K \). The other assertions are easily verified. \( \Box \)
We denote by $N_G H = NH = \{ n \in G \mid nHn^{-1} = H \}$ the normalizer of $H$ in $G$ and by $W_G(H) = WH$ the associated quotient group $NH/H$ (Weyl-group). Suppose $G$ is finite. Then $n^{-1}Hn \subset H$ implies $n^{-1}Hn = H$. Hence each endomorphism of $G/H$ is an automorphism. A $G$-map $f : G/H \to G/H$ has the form $gH \mapsto gnH$ for a uniquely determined coset $n_f \in WH$. The assignment $f \mapsto n_f^{-1}$ is an isomorphism $\text{Aut}_G(G/H) \cong WH$.

(1.3.2) Proposition. The right action of the automorphism group

$$G/H \times WH \to G/H, \quad (gH,nH) \mapsto gnH$$

is free. Hence for each $K \leq G$ the set $G/H^K$ carries a free $WH$-action and the cardinality $|G/H^K|$ is divisible by $|WH|$. We have $G/H^K = WH$. □

(1.3.3) Example. The assignment

$$\Psi_L : \text{Hom}_G(G/L,X) \to X^L, \quad \alpha \mapsto \alpha(eL)$$

is a bijection. The inverse sends $x \in X^L$ to $gL \mapsto gx$. We have

$$G/L^K = \{ sL \mid s^{-1}Ks \leq L \}.$$ Given $sL \in G/L^K$ then $R_s : G/K \to G/L, gK \mapsto gsL$ is the associated morphism. The diagram

$$\begin{array}{ccc}
\text{Hom}_G(G/L,X) & \xrightarrow{\Psi_L} & X^L \\
\downarrow{R_s} & & \downarrow{l_s} \\
\text{Hom}_G(G/K,X) & \xrightarrow{\Psi_K} & X^K
\end{array}$$

is commutative. We view the $\Psi_L$ as a natural isomorphism from the Hom-functor $\text{Hom}_G(-,X)$ to the fixed point functor. The left translation by $n \in NK$ maps $X^K$ into itself. In this way, $X^K$ becomes a $WK$-set.

Let $G$ be a finite group. The fixed point set $G/L^K$ is the set $\{ sL \mid s^{-1}Ks \leq L \}$. Let $A \leq L$ be $G$-conjugate to $K$. Consider the subset

$$G/L^K(A) = \{ tL \mid t^{-1}Kt \sim_L A \}.$$ The set $G/L^K$ has a left $N_G K$-action $(n,sL) \mapsto nsL$. The subsets $G/L^K(A)$ are $N_G K$-invariant.

(1.3.4) Proposition. Suppose $s^{-1}Ks = A$. The assignment

$$N_G(A)/N_L(A) \to G/L^K(A), \quad nN_L(A) \mapsto snL$$

is a bijection.
Proof. Since \( n^{-1}s^{-1}Ksn = n^{-1}An = A \leq L \) the element \( snL \) is contained in \( G/L^K \). The map is well-defined, because \( N_L(A) \leq A \). If \( snL = smL \), then \( m^{-1}n \in L \cap N_G(A) = N_L(A) \), and we see that the map is injective.

Suppose \( A \sim_L t^{-1}Kt \leq L \). Then there exists \( l \in L \) such that \( t^{-1}Kt = l^{-1}Al \), hence \( s^{-1}t^{-1} \in N_G(A) \), and \( tl = snL \). We see that the map is surjective.

We can rewrite this result in terms of the \( N_G(K) \)-action on \( G/L^K \). The subset \( G/L^K(A) \) is an \( N_G(K) \)-orbit; and the isotropy group at \( sL \) is \( sN_L(A)s^{-1} \).

The fixed point set \( G/L^K \) is the disjoint union of the \( G/L^K(A) \) where \( (A) \) runs over the \( L \)-conjugacy classes of the subgroups \( A \leq L \) which are \( G \)-conjugate to \( K \).

Since the homogeneous \( G \)-sets correspond to the subgroups of \( G \) we can consider a modified orbit category: The objects are the subgroups of \( G \) and the morphisms \( K \to L \) are the \( G \)-maps \( G/K \to G/L \). Since we are working with left actions we denote this category by \( \bullet \text{Or}(G) \). There is a similar category \( \text{Or}\bullet(G) \) where the morphisms \( K \to L \) are the \( G \)-maps \( K \setminus G \to L \setminus G \). If we assign to \( R : G/K \to G/L \) the map \( L_{s^{-1}} : K \setminus G \to L \setminus G, Kg \mapsto Ls^{-1}g \), then we obtain an isomorphism \( \text{Or}(G) \to \text{Or}\bullet(G) \).

The transport category \( \text{Tra}(G) \) of \( G \) has as object set the subgroups of \( G \), and the morphism set \( \text{Tra}(K,L) \) consists of the triples \( (K,L,s) \) with \( s \in G \) and \( sKs^{-1} \subset L \). We denote \( \text{Tra}(K,L) \) also as \( \{ s \in G \mid sKs^{-1} \leq L \} \) and pretend that the morphism sets are disjoint. Composition is defined by multiplication of group elements. In this context we work with the orbit category \( \text{Or}\bullet(G) \) of right homogeneous \( G \)-sets. We have a functor

\[
q : \text{Tra}(G) \to \text{Or}\bullet(G).
\]

It is the identity on objects and sends \((K,L,s)\) to \( l_s : K \setminus G \to L \setminus G, Kg \mapsto Lsg \).

The endomorphism sets in both categories are groups

\[
\text{Tra}(K,K) = NK, \quad \text{Or}\bullet(K,K) = WK.
\]

Via composition, \( \text{Tra}(K,L) \) carries a left action of \( NL = \text{Tra}(L,L) \) and a right action of \( NK = \text{Tra}(K,K) \). These actions commute. Similarly for the category \( \text{Or}\bullet(G) \). The functor \( q \) is surjective on Hom-sets and induces a bijection

\[
L/\text{Tra}(K,L) \cong \text{Or}\bullet(K,L).
\]

Let

\[
(K,L)_* = \{ A \mid A \leq L, A \sim_G K \}, \quad (K,L)^* = \{ B \mid K \leq B, B \sim_G L \}.
\]

(These sets can be empty.) We have bijections

\[
\text{Tra}(K,L)/NK \cong (K,L)_*, \quad s \cdot NK \mapsto sKs^{-1},
\]
1.3 The Orbit Category

\[ NL \setminus \text{Tr}(K, L) \cong (K, L)^*, \quad NL \cdot s \mapsto s^{-1} L s. \]

They imply the counting identities

\[ |(K, L)_*| \cdot |NK| = |(K, L)^*| \cdot |NL| = |G/L^K| \cdot |L|. \]  \hspace{1cm} (1.3)

The integers \( \zeta^*(K, L) = |(K, L)^*| \) and \( \zeta_*(K, L) = |(K, L)_*| \) depend only on the conjugacy classes of \( K \) and \( L \). The \( \text{Con}(G) \times \text{Con}(G) \)-matrices \( \zeta^* \) and \( \zeta_* \) have the property that their entries at \( (K), (L) \) are zero if \( (K) \not\leq (L) \), and the diagonal entries are 1. They are therefore invertible over \( \mathbb{Z} \), and their respective inverses \( \mu^* \) and \( \mu_* \) have similar properties. In order to see this, one solves the equation

\[ \sum_{(A)} \zeta^*(K, A) \mu^*(A, L) = \delta_{(K), (L)} \]  \hspace{1cm} (1.4)

inductively for \( \mu^*(K, L) \); the induction is over \( |\{(A) \mid (K) \leq (A) \leq (L)\}| \). The matrices \( \mu^* \) and \( \mu_* \) are called the Möbius-matrices of \( \text{Con}(G) \). Let \( N \) denote the diagonal matrix with entry \( |NK| \) at \( (K) \). Then (1.3) says

\[ \zeta_* N = N \zeta^*, \quad N \mu_* = \mu^* N. \]

Let \( G \) be abelian. Then

\[ \zeta^*(K, L) = \zeta_*(K, L) = \zeta(K, L) = 1, \quad \text{for } (K) \leq (L). \]

Hence also \( \mu^* = \mu_* = \mu \) in this case.

(1.3.5) Proposition. Let \( G \cong (\mathbb{Z}/p)^d \) be elementary abelian. Then \( \mu(1, G) = (-1)^d q^{d(d-1)/2} \).

Proof. The direct proof from the definition is a classical \( q \)-identity. For an indeterminate \( q \) we define the quantum number

\[ [n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}, \]

and the \( q \)-binomial coefficient

\[ [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad \binom{n}{a}_q = \frac{[n]_q!}{[a]_q! [n-a]_q!}. \]

With a further indeterminate \( z \) the following generalized binomial identity holds

\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j}_q q^{i(j-1)/2} z^j = \prod_{k=0}^{n-1} (1 - q^k z). \]  \hspace{1cm} (1.5)

A proof can be given by induction over \( n \), as in the case of the classical binomial identity. If \( q \) is a prime, then \( \binom{n}{j}_q \) is the number of \( j \)-dimensional subspaces of \( \mathbb{F}_q^n \). For \( z = 1 \) the identity (1.5) yields inductively the values of the \( \mu \)-function as claimed. \( \square \)
There are two more quotient categories of the transport category. The homomorphism category $\text{Sc}(G)$ has as morphism set the homomorphisms $K \to L$ which are of the form $k \mapsto gkg^{-1}$ for some $g \in G$. Two elements of $G$ define the same homomorphism if they differ by an element of the centralizer $ZK$ of $K$ in $G$. We can therefore identify the morphism set $\text{Sc}(K, L)$ with $\text{Tra}(K, L)/ZK$.

Finally we can combine the orbit category and the homomorphism category. In the category $\text{Sci}(G)$ we consider homomorphisms $K \to L$ up to inner automorphisms of $L$; thus the morphism set $\text{Sci}(K, L)$ can be identified with the double coset $L\backslash \text{Tra}(K, L)/ZK$.

Problems

1. Let $S$ be a $G$-set and $K \leq G$. We have a free left $WK$-action on $S_K$ via left translation. The inclusion $S_K \subset S(K)$ induces a bijection $S_K/WK \cong S(K)/G$. The map

$$G/K \times WK S_K \to S(K), \quad (gK, x) \mapsto gx$$

is a bijection of $G$-sets.

2. Suppose $D \leq G$ is cyclic. Then $|D||G/D| = |N_G A|$ if $(A)$ if $(D)$. Hence $\zeta(A, D) = 1$ for $(A) \leq (D)$. If $\mu: \mathbb{N} \to \mathbb{Z}$ denotes the classical Möbius-function, then

$$\mu(A, D) = \mu(D/A)$$

and

$$\mu^*(A, D) = NA/ND\mu(D/A).$$

The function $\mu$ is defined inductively by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ in the case that $n > 1$.

1.4 Möbius Inversion

We discuss in this section the Möbius matrices from a combinatorial viewpoint. Let $(S, \leq)$ be a finite partially ordered set (= poset). The Möbius-function of this poset is the function $\mu: S \times S \to \mathbb{Z}$ with the properties

$$\mu(x, x) = 1, \quad \sum_{y, x \leq y \leq z} \mu(x, y) = 0 \quad \text{for } x < z, \quad \mu(x, y) = 0 \quad \text{for } x \not\leq y.$$

These properties allow for an inductive computation of $\mu$. We use the Möbius-function for the Möbius-inversion: Let $f, g: S \to \mathbb{Z}$ be functions such that

$$g(x) = \sum_{y, x \leq y} f(y). \quad (1.6)$$

Then

$$f(x) = \sum_{y, x \leq y} \mu(x, y)g(y). \quad (1.7)$$
A more general combinatorial formalism uses the associative incidence algebra with unit \(I(S, \leq)\) of a poset. It consists of all functions \(f : S \times S \to \mathbb{Z}\) such that \(f(x, y) = 0\) if \(x \not\leq y\) with pointwise addition and multiplication

\[
(f \ast g)(x, y) = \sum_{z : x \leq z \leq y} f(x, z)g(z, y).
\]

(One can, more generally, define a similar algebra for functions into a commutative ring \(R\).) The unit element of this algebra is the Kronecker-delta

\[
\delta(x, y) = 1, \quad \text{for } x = y, \quad \delta(x, y) = 0, \quad \text{otherwise}.
\]

If we define the function \(\zeta\) by \(\zeta(x, y) = 1\) for \(x \leq y\), then the Möbius-function is the inverse of \(\zeta\) in this algebra \(\mu = \zeta^{-1}\). The group of functions \(\alpha : S \to \mathbb{Z}\) becomes a left module over the incidence algebra via \((f \ast \alpha)(x) = \sum_y f(x, y)\alpha(y)\).

We can now write (1.6) and (1.7) in the form

\[
\mu_{KL} = \sum_{K,B} \mu(K, B) \alpha_B.
\]

Moreover:

\[
\zeta_{KL} = \sum_{K,B} \zeta(K, B) \alpha_B.
\]

Let \(G\) be a finite group. We apply this to the poset \((\text{Sub}(G), \leq)\) and write \(\mu(1, H) = \mu(H)\), with the trivial group 1. Conjugation of subgroups yields an action of \(G\) on this poset by poset automorphisms. Let \(I^{\text{Con}}(G)\) denote the subalgebra of \(I(\text{Sub}(G), \leq)\) of invariant functions \(f(K, L) = f(gKg^{-1}, gLg^{-1})\).

We also have the poset \((\text{Con}(G), \leq)\) of conjugacy classes. For \(f \in I^{\text{Con}}(G)\) we define \(f_*(K, L) = \sum \{ f(A, L) \mid A \in (K, L)_s \}\) and \(f^*(K, L) = \sum \{ f(K, B) \mid B \in (K, L)^s \}\); see (1.1) and (1.2) for the notation. One verifies that \(f_*(K, L)\) and \(f^*(K, L)\) only depend on the conjugacy classes of \(K\) and \(L\).

**1.4 Möbius Inversion**

We have \(S_H = \coprod_{H \leq K} S_K\) and hence

\[
|S_H| = \sum_{H \leq K} \mu(H, K)|S_K|.
\]

Since \(S_H \cong G/H \times_{WH} S_H\), we see that the number \(m_H(S)\) of orbits of type \(H\) in \(S\) is given by

\[
m_H(S) = \frac{1}{|WH|} \sum_{H \leq K} \mu(H, K)|S_K|;
\]

note that \(|S_H/G| = |S_H/WH|\), and \(WH\) acts freely on \(S_H\). We rewrite this in terms of conjugacy classes:

\[
m_H(S) = \frac{1}{|WH|} \sum_{(K), H \leq (K)} \mu^*(H, K)|S_K|.
\]
1.5 The M"{o}bius Function

In this section we investigate the M"{o}bius-function by combinatorial methods.

(1.5.1) Lemma. Let \(1 \neq N \triangleleft G\) and \(N \leq K \leq G\). Then \(\sum_{X,N=K} \mu(X) = 0\).

Proof. By definition of \(\mu\), this holds for \(K = N\). We assume inductively, that the assertion holds for all proper subgroups \(Y\) of \(K\) which do not contain \(N\).

The computation

\[
0 = \sum_{X \leq K} \mu(K) = \sum_{X,XN=K} \mu(X) + \sum_{N \leq Y < K} \left( \sum_{XN=Y} \mu(X) \right) = \sum_{X,XN=K} \mu(X)
\]

yields the claim. \(\square\)

(1.5.2) Proposition. Let \(N \triangleleft G\) and let \(\text{Co}(G,N) = \{K \leq G \mid KN = G, K \cap N = 1\}\) be the set of complements of \(N\) in \(G\). Then

\[
\mu(G) = \mu(G/N) \cdot \sum_{K \in \text{Co}(G,N)} \mu(K,G).
\]

Proof. (An empty sum yields zero.) The assertion is trivial in the case that \(N = 1\); hence assume \(N > 1\). By (1.5.1)

\[
\mu(G) = - \sum_{X < G, XN = G} \mu(X).
\]

We use induction over the order of \(G\). This yields for the summand \(\mu(X)\)

\[
\mu(X) = \mu(X/X \cap N) \cdot \sum_{K \in \text{Co}(G,X \cap N)} \mu(K,X).
\]

Since \(XN = G\) we have \(G/N \cong X/X \cap N\). Therefore \(\mu(G)\) equals

\[
-\mu(G/N) \cdot \sum_{X < G, XN = G} \left( \sum_{K \in \text{Co}(G,X \cap N)} \mu(K,X) \right).
\]

One verifies that the following conditions (1) and (2) on \(X, K\) are equivalent:

(1) \(X \leq G, XN = G, K \in \text{Co}(X, X \cap N)\)

(2) \(K \leq X < G, K \in \text{Co}(G,N)\).

For (1) says \(K \leq X < G, K \cap N = 1; XN = G, K \cdot (X \cap N) = X\), and (2) says \(K \leq X < G, K \cap N = 1; K \cdot N = G\). In order to prove (1) \(\Rightarrow\) (2) we multiply the last equation in (1) with \(N\). In order to prove (2) \(\Rightarrow\) (1) we use the modular property of the subgroup lattice which says in general terms: For \(A, B, C \leq G\) and \(A \leq C\) the equality \(AB \cap C = A(B \cap C)\) holds. By definition of \(\mu\) we know

\[
\sum_{X,K \leq X \leq G} \mu(K,X) = -\mu(K,G).
\]

Now we put everything together and obtain the claim. \(\square\)
(1.5.3) **Proposition.** Let $G$ be a group with $\mu(G) \neq 0$. Let $N \leq M \lhd G$ and $N \triangleleft G$. Then $M/N$ has a complement in $G/N$.

**Proof.** By the previous note, $\mu(G/N) \neq 0$. Therefore it suffices to treat the case $N = 1$. But then, again by 1.5.2, $Co(G,M) \neq \emptyset$. \hfill $\Box$

The **Frattini-subgroup** $\Phi(G)$ of $G$ is the intersection of its maximal subgroups. The first assertion of the next note follows immediately from the definition of $\Phi(G)$. For the second one we use the fact that a maximal subgroup of a $p$-group is a normal subgroup of index $p$. See [?, III.3.2 and III.3.14].

(1.5.4) **Proposition.**
1. Let $N \triangleleft G$. Then there exists $H < G$ with $G = NH$ if and only if $N$ is not contained in $\Phi(G)$.
2. Let $G$ be a $p$-group. Then $G/\Phi(G)$ is elementary abelian, and $\Phi(G)$ is the smallest normal subgroup $N$ such that $G/N$ is elementary abelian. \hfill $\Box$

(1.5.5) **Corollary.** 1.5.3 and 1.5.4 imply:
1. If $\mu(G) \neq 0$. Then $\Phi(G) = 1$.
2. If $G$ is a $p$-group and $\mu(G) \neq 0$, then $G$ is elementary abelian. \hfill $\Box$

**Proof.** (1) Suppose $\mu(G) \neq 0$. Then we know from 1.5.3 that $\Phi(G)$ has a complement in $G$, and this is impossible, by 1.5.4(1), if $\Phi(G) \neq 1$.

(2) If $G$ is not elementary abelian, then $\Phi(G) \neq 1$, by 1.5.4(2), and therefore $\mu(G) = 0$. \hfill $\Box$

We now reprove 1.3.5.

(1.5.6) **Proposition.** Let $P$ be a $p$-group. Then $\mu^*(1,H) \neq 0$ if and only if $H \leq P$ is elementary abelian. If $H$ is elementary abelian of order $|H| = p^d$, then $\mu^*(1,H) = (-1)^dp^{d(d-1)/2}|P/NH|$.

**Proof.** We have just seen a proof of the first assertion. It remains to determine $\mu(G)$ for elementary abelian $G$. We induct over $|G|$. Suppose $A \leq G$, $|A| = p$, and $|G| = p^d$. Among the $(p^d - 1)/(p - 1) = b$ maximal subgroups exactly $(p^d - 1)/(p - 1) - 1 = a$ contain the subgroup $A$. Therefore $A$ has $b - a = p^{d-1}$ complements. By 1.5.2, $\mu(A) = -p^{d-1}\mu(G/A)$, since $\mu(K,G) = 1$ for a maximal subgroup $K$. \hfill $\Box$

**Problems**

1. The function $H \mapsto \mu(H, A_5)$ is displayed in the next table.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\mu(H, A_5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}/2$</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbb{Z}/3$</td>
<td>4</td>
</tr>
<tr>
<td>$\mathbb{Z}/5$</td>
<td>2</td>
</tr>
<tr>
<td>$D_2$</td>
<td>0</td>
</tr>
<tr>
<td>$D_3$</td>
<td>0</td>
</tr>
<tr>
<td>$D_5$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>1</td>
</tr>
</tbody>
</table>
1.6 One-dimensional Representations

We study in some detail the simplest type of representations, namely one-dimensional representations of finite groups \( G \) over the complex numbers; these are just the homomorphisms \( G \to \mathbb{C}^* \). These representations do not need much theory, and they will be used at various occasions, e.g., as input for the construction of more complicated representations (later called induced representations). The set \( X(G) = G^* \) of these homomorphisms becomes an abelian group with product \( (\alpha \cdot \beta)(g) = \alpha(g)\beta(g) \). A homomorphism \( \varphi: A \to B \) induces a homomorphism \( X(\varphi): X(B) \to X(A) \), \( \beta \mapsto \beta \circ \varphi \). In this manner \( X \) yields a contravariant functor from finite groups to finite abelian groups. The group \( X(G) \) will be called the character group of \( G \), and \( \alpha \in X(G) \) is a (linear) character of \( G \).

Since \( \mathbb{C}^* \) is abelian, a homomorphism \( \alpha: G \to \mathbb{C}^* \) maps the commutator subgroup \( [G,G] = G' \), generated by the commutators \( uvu^{-1}v^{-1} \), to 1 and induces \( \pi: G/[G,G] \to \mathbb{C}^* \). The factor group \( G/[G,G] \) is abelian and it is called the abelianized quotient \( G^{ab} \) of \( G \). For the cyclic group \( \mathbb{C}_m = \langle c | c^m = 1 \rangle \) the character group \( X(C_m) \) is the cyclic group of order \( m \) generated by \( \rho: c \mapsto \exp(2\pi i/m) \).

Let \( H \) be a normal subgroup of \( G \). Then the group \( G \) acts as a group of automorphisms on \( X(H) \) by \( (g \cdot \gamma)(h) = \gamma(ghg^{-1}) \). If \( \gamma \) is the restriction of a homomorphism \( \alpha \in X(G) \), then \( g \cdot \gamma = \gamma \). Therefore the restriction homomorphism \( X(G) \to X(H) \) has an image in the fixed point group \( X(H)^G \); its elements are called \( G \)-invariant. Note that the \( G \)-action on \( X(H) \) factors over \( G/H \).

Let \( G \) be the semi-direct product of the normal subgroup \( A \) and a group \( P \), i.e. \( G = AP \) and \( A \cap P = 1 \). Let

\[ \Gamma: X(G) \to X(A)^P \times X(P) \]

be the product of the restriction homomorphisms.

---

1 Recall the categorical notion: sum in the category of abelian groups.
(1.6.1) Proposition. \( \Gamma \) is an isomorphism.

Proof. Since \( \gamma \in X(G) \) is determined by the restrictions to \( A \) and \( P \), the map \( \Gamma \) is injective. Given \( \alpha \in X(A)^P \), i.e. \( \alpha(a) = \alpha(axx^{-1}) \) for \( a \in A, x \in P \), and \( \beta \in X(P) \). Define a map \( \gamma: G \to \mathbb{C}^* \) by \( \gamma(ax) = \alpha(a)\beta(x) \). We verify that \( \gamma \) is a homomorphism

\[
\begin{align*}
\gamma(axa_1a_2) &= \gamma(axa_1x^{-1}x_1) \\
&= \alpha(axa_1x^{-1})\beta(x_1) \\
&= \alpha(a)\alpha(xax^{-1})\beta(x_1) \\
&= \alpha(a)\beta(x)\alpha(a_1)\beta(x_1) \\
&= \gamma(ax)\gamma(a_1a_2).
\end{align*}
\]

By construction, \( \Gamma(\gamma) = (\alpha, \beta) \).

A homomorphism \( \alpha \in X(A) \) is \( P \)-invariant if and only if it vanishes on the normal subgroup \( A_P \) generated by the elements \( axa^{-1}x^{-1} \) for \( a \in A \) and \( x \in P \). Thus the quotient map \( \pi: A \to A/A_P \) induces an isomorphism \( X(A/A_P) \to X(A)^P \).

(1.6.2) Proposition. Let the group \( P \) act on the abelian group \( A \) by automorphisms \( x, a \mapsto x \circ a \). Suppose \( |A|, |P| = 1 \). Let \( A_P \leq A \) denote the subgroup generated by the elements \( a \cdot (x \circ a)^{-1} \). Then the inclusion \( i: A^P \hookrightarrow A/A_P \) is an isomorphism.

Proof. For \( a \in A \) set \( \mu(a) = \prod_{x \in P(x \circ a)} \). Then \( \mu(a) \in A_P \), and for \( a \in A^P \) we have \( \mu(a) = a^{P|} \). Since \( |P| \) is prime to the order of \( A \), the map \( a \mapsto a^{P|} \) is an automorphism of \( A \) and \( A^P \). The group \( A_P \) is contained in the kernel of \( \mu \), since, by construction, \( \mu(y \circ a) = \mu(a) \) for \( a \in A \) and \( x \in P \). Hence we obtain an induced map \( \nu: A/A_P \to A^P \), and \( \nu \circ i \) is an isomorphism. On the other hand \( \nu(a) = a^{P|} \prod_{i}(x \circ a)^{a^{-1}} \), and this shows that \( i \circ \nu \) is an isomorphism too.

As a consequence of (1.6.1) and (1.6.2) we obtain the next result which will later be used in the proof of the Brauer induction theorem.

(1.6.3) Proposition. Let \( G = AP \) be the semi-direct product of the abelian subgroup \( A \) by \( P \). Suppose \( |A|, |P| = 1 \). Then the restriction \( X(G) \to X(A^P \times P) \) is an isomorphism.

We continue the study of one-dimensional representations and demonstrate their use in group theory. Let \( H \leq G \) and \( \alpha \in X(H) \). We associate to \( \alpha \) an element \( m_P^G \alpha \in X(G) \). For this purpose we choose a representative system
g_1, \ldots, g_r \text{ of } G/H. \text{ For each } g \in G \text{ we have } gg_i = g_{\sigma(i)}h_i \text{ with a permutation } 
abla \sigma \in S_r \text{ and certain } h_i \in H. \text{ We set }

(m_H^G \alpha)(g) = \prod_{i=1}^r \alpha(g_{\sigma(i)}g h_i) = \prod_{i=1}^r \alpha(h_i).

One verifies that \(m_H^G \alpha\) is a well-defined homomorphism and that, moreover, \(m_H^G \colon X(H) \to X(G)\) is a homomorphism. We call it \textbf{multiplicative induction}.

We use this construction to deal with the question: Given \(\alpha \in X(H)\), when does there exist an extension \(\beta \in X(G)\) such that \(\beta|H = \alpha\)? Suppose it exists. Then for \(h \in H\), \(u \in G\) and \(g = uh^{-1}\) we have \(\beta(g) = \beta(h)\). Thus a necessary condition for the existence of \(\beta\) is that \(\alpha\) is trivial on the subgroup \(H_0\) generated by \(\{xy^{-1} \mid x \sim_H y, x, y \in H\}\).

\textbf{(1.6.4) Proposition.} Suppose \(H\) and \(G/H\) have coprime order. Then an extension \(\beta\) exists if and only if \(\alpha\) vanishes on \(H_0\).

\textit{Proof.} Let \(\alpha\) have the stated property. We compute \(m_H^G \alpha\)(h) for \(h \in H\). For this purpose we make a special choice of the coset representatives: The cyclic group \((h)\) acts on \(G/H\): let \(gh, hgh, \ldots, h^t g H\) be an orbit, and suppose \(h^tg = gh\). Then this orbit contributes \(\alpha(h) = \alpha(g^{-1}h^tg) = \alpha(h)^t\) to the product in the definition of \(m_H^G \alpha\). Altogether we obtain \(\text{res}_H^G m_H^G \alpha = \alpha|^{G/H}\). Hence if \(H \text{ and } G/H \text{ have coprime order, then } \text{res}_H^G m_H^G \alpha\) is an automorphism because \(|X(H)| = |G/H|\). Therefore there exists an extension. \(\square\)

\textbf{(1.6.5) Proposition.} Let \(H\) be a Sylow \(p\)-subgroup of \(G\). Then \(H_0 = H \cap G'\).

\textit{Proof.} If \(g = g^{-1}xg\), then \(xy^{-1} = xg^{-1}x^{-1}g\), so that \(P^p \leq P_0 \leq P \cap G'\). It remains to show \(P \cap G' \leq P_0\). Given \(x \in P \setminus P_0\), there exists \(\lambda \in X(P)\) such that \(\lambda(x) \neq 1\) with trivial \(\lambda|P_0\). By the previous proposition, \(\lambda\) has an extension \(\theta : G \to \mathbb{C}^*\). Since \(\theta(x) \neq 1\), we see that \(x \notin P \cap G'\). \(\square\)

The previous considerations lead to a simple proof of the so-called \textbf{normal complement theorem}.

\textbf{(1.6.6) Proposition.} Let \(G(p)\) be an abelian Sylow \(p\)-group of \(G\) and assume that \(NG(p) = G(p)\). Then there exists a normal subgroup \(H < G\) such that \(N \cap G(p) = 1\).

\textit{Proof.} The quotient \(G/G'\) has Sylow group \(G(p)\) if \(G' \cap P = P_0 = 1\). This means: Suppose \(x, y \in P\) are conjugate in \(G\), then \(x = y\). This is a consequence of the next lemma. Since \(G/G'\) is abelian, there exists a complement of \(G(p)\), and the pre-image in \(G\) is the required complement. \(\square\)

\textbf{(1.6.7) Lemma.} Let \(G\) have abelian Sylow \(p\)-group \(P\). Suppose \(x, y \in P\) are conjugate in \(G\). Then they are conjugate in \(NGP\).
Proof. Let \( y = gxg^{-1} \). Since \( P \) is abelian, \( P \) is a subgroup of the centralizer \( C_G(y) \) of \( y \) in \( G \), moreover \( g(C_G(x))g^{-1} = C_G(y) \). Hence \( gPg^{-1} \) and \( P \) are Sylow groups of \( C_G(y) \). Therefore there exists \( n \in C_G(y) \) such that \( ngPg^{-1}n^{-1} = P \). Hence \( ng \in NG(P) \) and \( y = ny^{-1} = ngxg^{-1}n^{-1} \).

Problems

1. Let \( B \) be a subgroup of the finite abelian group \( A \). Show that for each \( a \in A \setminus B \) there exists \( \alpha \in X(A) \) with \( \alpha(a) \neq 1 \).
2. The isomorphism between \( G \) and \( G^* \) is not natural, but there exists a canonical and natural isomorphism \( G \to X(X(G)) \), \( G \) abelian. (This is analogous to the double dual of finite dimensional vector spaces.)
3. Let \( 1 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 1 \) be an exact sequence of finite abelian groups. Then the functor \( X \) transforms it into an exact sequence. Exactness at \( X(B) \) and \( X(C) \) is formal; for the exactness at \( X(A) \) one can use the knowledge of the order of this group.

1.7 Representations as Modules

The vector space \( KG \) has more structure than just carrying the left and right regular representation.

There is a bilinear map \( KG \times KG \to KG \) which extends the group multiplication \( (g, h) \mapsto gh \) of the basis elements. This bilinear map defines on \( KG \) the structure of an associative algebra with unit. This algebra is called the **group algebra** \( KG \) of \( G \) over \( K \). The multiplication in the group algebra is therefore defined by the formula

\[
(\sum_{g \in G} \lambda(g)g) \cdot (\sum_{h \in G} \mu(h)h) = \sum_{g,h} \lambda(g)\mu(h)gh = \sum_{u \in G} \gamma(u)u
\]

with \( \gamma(u) = \sum_{g \in G} \lambda(g)\mu(g^{-1}u) \). Another model for the group algebra is the vector space \( C(G, K) \) of functions \( G \to K \) with **convolution product**

\[
(\alpha * \beta)(u) = \sum_{g \in G} \alpha(g^{-1})\beta(u^{-1}g).
\]

The assignment \( C(G, K) \to KG, \varphi \mapsto \sum_{g} \varphi(g^{-1})g \) is an isomorphism of algebras. Under this isomorphism the natural left-right action on \( C(G, K) \), given by

\[
(g \cdot \varphi \cdot h)(x) = \varphi(hxg),
\]

corresponds to the left-right action on \( KG \).
(1.7.1) Example. The group algebra of the cyclic group $C_n = \langle x \mid x^n = 1 \rangle$ is the quotient $K[x]/(x^n - 1)$ of the polynomial algebra $K[x]$ by the principal ideal $(x^n - 1)$.

We now come to the third form of a representation, that of a module over the group algebra. Let $V$ be a $KG$-representation. The bilinear map

$$KG \times V \to V, \quad (\sum g \lambda(g)g, v) \mapsto \sum g \lambda(g)(g \cdot v)$$

is the structure of a unital $KG$-module on the vector space $V$. The element $\sum g \lambda(g)g \in KG$ acts on $V$ as the linear combination $\sum g \lambda(g)g$. A morphism $V \to W$ of representations becomes a $KG$-linear map. Conversely, given a $KG$-module $M$ we obtain a representation on $M$ by defining $l_g$ as the scalar multiplication by $g \in KG$ in the module. In this manner, the category $KG$-Rep of finite-dimensional $KG$-representations becomes the category $KG$-Mod of left $KG$-modules which are finite-dimensional as vector spaces. Direct sums correspond in these categories. A module $M$ over an algebra $A$ is irreducible, if it has no submodules different from 0 and $M$.

The viewpoint of modules allows for an algebraic construction of representations. Consider $KG$ as a left module over itself. Then a left ideal is a representation. A non-zero left ideal yields an irreducible module, if it is a minimal left ideal with respect to inclusion. Let $M$ be an irreducible $KG$-module and $0 \neq x \in M$. Then $KG \to M, \lambda \mapsto \lambda x$ is $KG$-linear; its kernel $I$ is a left ideal and the induced map $KG/I \to M$ an isomorphism, since $M$ is irreducible; the ideal $I$ is then a maximal ideal.

(1.7.2) Example. The maximal ideals in the group algebra $KC_n = K[x]/(x^n - 1)$ correspond to principal ideals $(q) \subset K[x]$ where $q$ is an irreducible factor of $x^n - 1$. If $K$ is a splitting field for $x^n - 1$, then the irreducible factors are linear, and irreducible representations one-dimensional. Over $\mathbb{Q}$, the polynomial is the product $\prod_{d \mid n} \Phi_d(x)$ of the irreducible cyclotomic polynomials $\Phi_d$. The complex roots of $\Phi_d$ are the primitive $d$-th roots of unity. As an example

$$x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1).$$

The representation on $\mathbb{Q}[x]/(x^2 - x + 1)$ is given in the basis $1, x$ by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus we know that this matrix has order 6; this can, of course, be checked by a calculation. One the other hand, it is a nontrivial task to find matrices in $SL_2(\mathbb{Z})$ of order 6.
1.8 Linear Algebra of Representations

Standard constructions of linear algebra may be used to obtain new representations from old ones. We begin with direct sums.

Let \( V_1, \ldots, V_r \) be vector spaces over \( K \). Their (external) direct sum \( V_1 \oplus \cdots \oplus V_r \) consists of all \( r \)-tuples \((v_1, \ldots, v_r)\), \( v_j \in V_j \) with component-wise addition and scalar multiplication. If the \( V_j \) are subspaces of a vector space \( V \), we say, \( V \) is the (internal) direct sum of these subspaces, if each \( v \in V \) has a unique presentation of the form \( v = \sum_{j=1}^r v_j \) with \( v_j \in V_j \). We also use the notation \( V = \bigoplus_{j=1}^r V_j \), because \( V \) is canonically isomorphic to the external direct sum of the \( V_j \). A subspace \( U \) of \( W \) is a direct summand, if there exists a complementary subspace \( V \), i.e., a subspace \( V \) such that \( U \oplus V = W \).

Let \( (V_j \mid j \in J) \) be a family of subspaces of \( V \). The sum \( \sum_{j \in J} V_j \) is the subspace of \( V \) generated by the \( V_j \). It is the smallest subspace containing the \( V_j \) and consists of the elements which are sums of elements in the various \( V_j \). We use the following fact from linear algebra.

\[ \text{(1.8.1) Proposition. Let } V_1, \ldots, V_n \text{ be subspaces of } V. \text{ The following are equivalent:} \]

1. \( V \) is the internal direct sum of the \( V_j \).
2. \( V \) is the sum of the \( V_j \), and \( V_j \cap \sum_{i \neq j} V_i = \{0\} \) for all \( j \).

We now apply these concepts to representations. We use two simple observations. If \( (V_j \mid j \in J) \) are sub-representations of \( V \), then their sum is again a sub-representation. The direct sum \( U \oplus V \) of representations becomes a representation with respect to the component-wise group action \( g \cdot (u, v) = (g \cdot u, g \cdot v) \). Similarly for an arbitrary number of summands. This defines the direct sum of representations. If \( g \mapsto A(g) \) and \( g \mapsto B(g) \) are matrix representations associated to \( U \) and \( V \), then the block matrices

\[
\begin{pmatrix}
A(g) & 0 \\
0 & B(g)
\end{pmatrix}
\]

yield a matrix representation for \( U \oplus V \). A representation is called indecomposable, if it is not the direct sum of non-zero sub-representations. An irreducible representation is clearly indecomposible, but the converse does not hold in general.

\[ \text{(1.8.2) Example. In Example 1.1.4 we defined two sub-representations } T_n, D \text{ of the permutation representation of } S_n \text{ on } K^n. \text{ Given } (x_1, \ldots, x_n) \in K^n \text{ write } x = n^{-1} \sum_j x_j. \text{ Then } (x_1 - x, \ldots, x_n - x) \in T_n \text{ and } (x, \ldots, x) \in D. \text{ Hence } T_n + D = K^n. \text{ The intersection } T_n \cap D \text{ consists of the } (y, \ldots, y) \text{ with } ny = 0. \text{ This implies } y = 0. \text{ Hence } T_n \oplus D = K^n. \text{ But note: This argument requires that } n^{-1} \text{ makes sense in } K, \text{ i.e., the characteristic of } K \text{ does not divide } n. \]
If \( n = 2 \) and \( K = \mathbb{F}_2 \) is the field with two elements, then \( V = W! \) The regular representation is not irreducible, because it has a one-dimensional fixed point set. If this fixed point set had a complement it would be a one-dimensional representation, hence a trivial representation. Therefore the regular representation is indecomposable.

Here is another result from linear algebra.

\[ (1.8.3) \text{ Proposition.} \] A sub-representation \( W \) of \( V \) is a direct factor if and only if there exists a projection morphism \( q: V \rightarrow V \) with image \( W \). A projection is a morphism \( q \) such that \( q \circ q = q \). If \( q \) is a projection, then \( V \) is the direct sum of the image and the kernel of \( q \).

Let \( V \) and \( W \) be representations of \( G \). The tensor product representation \( V \otimes_K W \) has the action \( g(v \otimes w) = gv \otimes gw \). If \( v_1, \ldots, v_n \) is a basis of \( V \) and \( w_1, \ldots, w_m \) is a basis of \( W \), then the \( v_i \otimes w_k \) form a basis of \( V \otimes K \). The map \( V \times W \rightarrow V \otimes W, (v,w) \mapsto v \otimes w \) is bilinear. If \( g \) acts on \( V \) and \( W \) via matrices \( (r_{ij}) \) and \( (s_{kl}) \), then \( g \) acts on \( V \otimes W \) via the matrix \( (r_{ij}s_{kl}) \) whose entry in the \( (i,k) \)-th row and \( (j,l) \)-th column is \( r_{ij}s_{kl} \). More explicitly, if \( gv_j = \sum_i r_{ij}v_i \) and \( gw_l = \sum_k s_{kl}w_k \), then

\[ g(v_j \otimes w_l) = \sum_{i,k} r_{ij}s_{kl}v_i \otimes w_k. \]

If \( V \) is one-dimensional and given by a homomorphism \( \alpha: G \rightarrow K^* \), then we simply multiply the matrix \( (s_{kl}) \) with \( \alpha(g) \) in order to obtain the tensor product.

Let \( V \) and \( W \) be \( G \)-representations. We have a \( G \)-action on the vector space \( \text{Hom}(V,W) \) of \( K \)-linear maps, given by \( (g \cdot \varphi)(v) = g\varphi(g^{-1}v) \). The fixed point set is \( \text{Hom}(V,W)^G = \text{Hom}_G(V,W) \). When \( W = K \) is the trivial representation we obtain the dual representation \( V^* = \text{Hom}(V,K) \) of \( V \). If \( g \mapsto A(g) \) is the matrix representation of \( V \) with respect to a basis, then \( g \mapsto inverse(\text{transpose}) \) (inverse of the transpose) is the matrix representation of \( V^* \) with respect to the dual basis.

\[ (1.8.4) \text{ Note.} \] There is a canonical isomorphism

\[ V^* \otimes W \overset{\cong}{\rightarrow} \text{Hom}(V,W), \quad \varphi \otimes w \mapsto (u \mapsto \varphi(u)w). \]

One verifies that it is a morphism of \( G \)-representations.

In some of the constructions one can also use representations for different groups. Let \( V \) be a \( G \)-representation and \( W \) an \( H \)-representation. Then \( V \otimes W \) becomes a \( G \times H \)-representation via \( (g,h)(v \otimes w) = gv \otimes hw \). Similarly, we have a \( G \times H \)-action on \( \text{Hom}(V,W) \) defined as \( ((g,h) \cdot \psi)(v) = h\psi(g^{-1}v) \). With these actions, \( 1.8.4 \) is an isomorphism of \( G \times H \)-representations.
(1.8.5) **Example.** Let $S$ and $T$ be finite $G$-sets. There are canonical isomorphisms

$$K(S \sqcup T) \cong K(S) \oplus K(T), \quad K(S \times T) \cong K(S) \otimes K(T), \quad K(S)^* \cong K(S).$$

They are induced by a $G$-equivariant bijection of the canonical bases. We combine with 1.8.4 and obtain

$$\text{Hom}(K(S), K(T)) \cong K(S)^* \otimes K(T) \cong K(S) \otimes K(T) \cong K(S \times T).$$

Together with 1.2.2 we get

$$\dim_K \text{Hom}_G(K(S), K(T)) = |(S \times T)/G|.$$

The representation 1.1.4 of $S_n$ on $K^n$ by permutation of coordinates (made into a left representation by inversion) is isomorphic to $K(S_n/S_n-1)$ where $S_n-1$ is the subgroup of $S_n$ which fixes $1 \in \{1, \ldots, n\}$. The action of $S_n-1$ on $S_n/S_n-1$ has two orbits, of length 1 and $n-1$. Hence $\text{Hom}_{S_n}(K^n, K^n)$ is two-dimensional.

## 1.9 Semi-simple Representations

The topic of this section is the decomposition of a representation into a direct sum of sub-representations.

We begin with a simple and typical example. Let $\alpha: G \to K^*$ be a homomorphism. Consider $x_\alpha = \sum_{g \in G} \alpha(g^{-1})g \in KG$. The computation

$$h \cdot x_\alpha = \sum_{g \in G} \alpha(g^{-1})hg = \sum_{g \in G} \alpha(h)\alpha(g^{-1}h^{-1})hg = \alpha(h)x_\alpha$$

shows that $x_\alpha$ spans a one-dimensional sub-representation $V(\alpha)$ of the regular representation. Let $K = \mathbb{C}$ and $G = C_n = \langle a \mid a^n = 1 \rangle$ the cyclic group.

There are $n$ different homomorphisms $\alpha(j): C_n \to \mathbb{C}^*$, $1 \leq j \leq n$. The vectors $x_{\alpha(j)}$ are different eigenvectors of $l_a$. Therefore we have a decomposition $\mathbb{C}C_n = \oplus_{j=1}^n V(\alpha(j))$ into one-dimensional representations. A similar decomposition exists for finite abelian groups $G$, since we still have $|G|$ homomorphisms $G \to \mathbb{C}^*$. Our aim is to find analogous decompositions for general finite groups.

(1.9.1) **Theorem.** Let $V$ be the sum of irreducible representations (U\_j \mid j \in J) and let $U$ be a sub-representation. Then there exist a finite subset $E \subset J$ such that $V$ is the direct sum of $U$ and the $U_j, j \in E$.

**Proof.** If $W \neq V$ is any sub-representation, then there exists $k \in J$ such that $V_k \not\subset W$, since $V$ is the sum of the $V_j$. Then $V_k \cap W = 0$, and $W + V_k = W \oplus V_k$.

If now $E \subset J$ is a maximal subset such that the sum $W$ of $U$ and the $V_j, j \in E$ is direct, then necessarily $W = V$. \[\square\]
(1.9.2) Theorem. The following assertions about a representation $M$ are equivalent:

1. $M$ is a direct sum of irreducible sub-representations.
2. $M$ is a sum of irreducible sub-representations.
3. Each sub-representation is a direct summand.

Proof. (1) $\Rightarrow$ (2) as special case; and (2) $\Rightarrow$ (3) is a special case of 1.9.1.

(3) $\Rightarrow$ (1). Let $\{M_1, \ldots, M_n\}$ be a set of irreducible sub-representations such that their sum $N$ is the direct sum of the $M_j$. If $N \neq M$ then, by hypothesis, there exists a sub-representation $L$ such that $M = N \oplus L$. Each sub-representation contains an irreducible one. If $M_{n+1} \subset L$ is irreducible, then the sum of the $\{M_1, \ldots, M_{n+1}\}$ is direct.

A representation is called **semi-simple** or **completely reducible** if it has one of the properties (1)-(3) in 1.9.2.

(1.9.3) Proposition. Sub-representations and quotient representations of semi-simple representations are semi-simple.

Proof. Let $M$ be semi-simple and $F \subset N \subset M$ sub-representations. A projection $M \to M$ with image $F$ restricts to a projection $N \to N$ with image $F$. Hence $F$ is a direct summand in $N$.

Suppose $N \oplus P = M$; then the quotient $M/N \cong P$ is semi-simple.

(1.9.4) Proposition. Let $V$ be the sum of irreducible sub-representations $(V_j \mid J)$. Then each irreducible sub-representation $W$ is isomorphic to some $V_j$.

Proof. There exists a surjective homomorphism $\beta: V \to W$, by 1.8.3 and 1.9.2. If $W$ were not isomorphic to some $V_j$, then the restriction of $\beta$ to each $V_j$ would be zero, by Schur’s lemma, hence $\beta$ would be the zero morphism.

We write

$$\langle U, V \rangle = \dim_K \text{Hom}_G(U, V)$$

for $G$-representations $U$ and $V$. This integer depends only on the isomorphism classes of $U$ and $V$. Note the additivity $\langle U_1 \oplus U_2, V \rangle = \langle U_1, V \rangle + \langle U_2, V \rangle$, and similarly for the second argument.

(1.9.5) Proposition. Suppose $V = V_1 \oplus \cdots \oplus V_r$ is a direct sum of irreducible representations $V_j$. Let $W$ be any irreducible representation and denote by $n(W,V)$ the number of $V_j$ which are isomorphic to $W$. Then

$$\langle W, W \rangle n(W,V) = \langle W, V \rangle = \langle V, W \rangle.$$

Therefore $n(W,V)$ is independent of the decomposition of $V$ into irreducibles.
Proof. For a direct sum as above, we have a canonical isomorphism
\[ \text{Hom}_G(W, V) \cong \prod_{j=1}^r \text{Hom}_G(W, V_j). \]
This expresses the fact that a morphism \( W \to V \) is nothing else but an \( r \)-tuple of morphisms \( W \to V_j \). The assertion is now a direct consequence of Schur’s lemma. For the second assertion we use the canonical isomorphism \( \text{Hom}_G(V, W) \cong \prod_{j} \text{Hom}_G(V_j, W) \). (In conceptual terms: We are using the fact that \( \bigoplus_j V_j \) is the sum and the product of the \( V_j \) in the category of representations.)

We call the integer \( n(W, V) \) in 1.9.5 the multiplicity of the irreducible representation \( W \) in the semi-simple representation \( V \). We say \( W \) occurs in \( V \) or is contained in \( V \) if \( n(W, V) \neq 0 \). In fact, if \( n(W, V) \neq 0 \), then \( V \) has a sub-representation which is isomorphic to \( W \): take a non-zero morphism \( W \to V \) and apply Schur’s lemma. The irreducible representation \( W \) appears in \( V \) if and only if \( \text{Hom}_G(W, V) \) or \( \text{Hom}_G(V, W) \) is non-zero.

Let \( W \) be irreducible and denote by \( V(W) \) the sum of the irreducible sub-representations of \( V \) which are isomorphic to \( W \). We call \( V(W) \) the \( W \)-isotypical part of \( V \), if \( V(W) \neq 0 \), and the decomposition in 1.9.6 is the isotypical decomposition of \( V \). Let \( I = \text{Irr}(G; K) \) denote a complete set of pairwise non-isomorphic irreducible representations of \( G \) over \( K \).

(1.9.6) Theorem. A semi-simple representation \( V \) is the direct sum of its isotypical parts.

Proof. Since \( V \) is semi-simple it is the direct sum of irreducible sub-representations and therefore the sum of its isotypical parts. Let \( A \in I \) and let \( Z \) be the sum of the \( V(B) \), \( B \in I, B \neq A \). We refer to 1.8.1 and have to show: \( V(A) \cap Z = 0 \). Suppose this is not the case. Then the intersection would contain an irreducible sub-representation, and by 1.9.4 it would be isomorphic to \( A \) and to some \( B \neq A \). Contradiction. \( \square \)

1.10 The Regular Representation

We now consider \( KG \) as left and right regular representation. For each representation \( U \) the vector space \( \text{Hom}_G(KG, U) \) becomes a left \( G \)-representation via \( (g \cdot \varphi)(x) = \varphi(x \cdot g) \).

(1.10.1) Lemma. The evaluation \( \text{Hom}_G(KG, U) \to U, \varphi \mapsto \varphi(e) \) is an isomorphism of representations.
Proof. We use the fact that $KG$ is a free $KG$-module with basis $e$. It is verified from the definitions that the evaluation is a morphism. Clearly, a morphism $KG \to U$ is determined by its value at $e$, and this value can be any prescribed element of $U$.

\textbf{(1.10.2) Theorem.} Suppose the left regular representation is semi-simple. Then each irreducible representation $U$ appears in $KG$ with multiplicity $n_U = \langle U, U \rangle^{-1} \dim_K U$.

Proof. Since $U \cong \Hom_G(KG, U)$ is non-zero, each irreducible representation $U$ appears in $KG$, see the remarks after 1.9.5. Suppose $KG \cong \bigoplus_{W \in I} n_W W$ where $n_W W$ denotes the direct sum of $n_W$ copies of $W$. Then
\[ \dim U = \langle KG, U \rangle = \sum_{W \in I} n_W \langle W, U \rangle = n_U \langle U, U \rangle, \]
the latter by Schur’s lemma.

\textbf{(1.10.3) Proposition.} Suppose the left regular representation is semi-simple.
\begin{enumerate}
\item The number of isomorphism classes of irreducible representations is finite.
\item $|G| = \sum_{V \in I} \langle V, V \rangle^{-1} (\dim V)^2$.
\item If $K$ is algebraically closed, then $|G| = \sum_{V \in I} (\dim V)^2$.
\end{enumerate}

Proof. (1) is a corollary of 1.10.2

(2) Let $KG \cong \bigoplus_{W \in I} n_W W$. We insert the values of $n_W$ obtained in 1.10.2

(3) If the field $K$ is algebraically closed then, by Schur’s lemma, $\langle V, V \rangle = 1$ for an irreducible representation $V$.

Part (3) of 1.10.3 gives us a method to decide whether a given set of pairwise non-isomorphic irreducible representations is complete. If $G$ is abelian then irreducible representations over $\mathbb{C}$ are 1-dimensional. By 1.10.3 we see that there are $|G|$ non-isomorphic such representations; we know this, of course, from a direct elementary argument.

\textbf{(1.10.4) Proposition.} A finite group $G$ is abelian if and only if the irreducible complex representations are one-dimensional.

Proof. A one-dimensional complex representation is given by a homomorphism $G \to \mathbb{C}^*$. The regular representation is faithful. If the regular representation is a sum of one-dimensional representations, then $G$ has an injective homomorphism into an abelian group. The reversed implication was proved in 1.1.3

There remains the question: When is $KG$ semi-simple? Recall some elementary algebra. For $n \in \mathbb{N}$ and $x \in K$, an expression $nx$ stands for an $n$-fold sum $x + \cdots + x$. A relation $nx = 1$ exists in $K$ if and only if either $K$ has characteristic zero or the characteristic $p > 0$ of $K$ does not divide $n$. In this case we say, $n$ is invertible in $K$. We denote this inverse as usual by $n^{-1}$.\[\begin{array}{c}
\end{array}\]
(1.10.5) Proposition. If $KG$ is semi-simple, then $|G|$ is invertible in $K$.

Proof. Suppose $KG$ is semi-simple. Then the fixed set $F = \{ \lambda \Sigma \mid \lambda \in K, \Sigma = \sum_{g \in G} g \}$ is a sub-representation. Hence there exists a projection $p: KG \to F$. Since $p$ is $G$-equivariant, it is determined by the value $p(e)$, say $p(e) = \mu \Sigma$. Since $p$ is a projection we obtain $\Sigma = p(\Sigma) = \sum_{g \in G} p(e) = \sum_{g \in G} \mu \Sigma = |G| \mu \Sigma$. Thus we have shown: If $KG$ is semi-simple, then $|G|$ is invertible in $K$. \qed

(1.10.6) Theorem (Maschke). Suppose $|G|$ is invertible in $K$. Then $G$-representations are semi-simple.

Proof. We show that each sub-representation $W$ of a representation $V$ is a direct summand (see 1.9.2). There certainly exists a $K$-linear projection $p: V \to V$ with image $W$. We make it equivariant by an averaging process. Namely we define

$$q(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1}p(gv).$$

At this point we use the fact that $|G|^{-1}$ makes sense in $K$. By construction, $q$ is $K$-linear as a linear combination of linear maps. For $h \in G$ we compute

$$q(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}p(ghv) = \frac{1}{|G|} h \sum_{g \in G} h g^{-1}p(ghv) = hq(v),$$

and this verifies the equivariance. By hypothesis, $p(w) = w$ for $w \in W$, hence $p(gw) = gw$ and therefore $q(w) = w$. The values $q(v)$ are contained in $W$, hence $W = q(V)$ and $q^2 = q$. \qed

(1.10.7) Proposition. Suppose $V$ is semi-simple. Then $\langle V, V \rangle = 1$ implies that $V$ is irreducible.

Proof. Decompose into irreducibles $V = \sum n_W W$. Then $1 = \langle V, V \rangle = \sum n_W^2 \langle W, W \rangle$, by Schur’s lemma. Hence one of the $n_W$ is 1 and the others are 0. \qed

Assume that $|G|$ is invertible in $K$. In order that $\ref{1.10.3}$ holds, it is necessary to assume that $\langle V, V \rangle = 1$ for each $V \in I$. If $K \subset L$ is a field extension, then we can view a $K$-representation as an $L$-representation (just take the same matrices). However, an irreducible representation over $K$ may become reducible over a larger field. This already happens for cyclic groups, as we have seen in the first section. If $KG$ is semi-simple, then also $LG$. If the relation $\ref{1.10.3}$ holds for $K$-representations, then it also holds for $L$-representations. The relation $\ref{1.10.3}$ is equivalent to $\langle V, V \rangle = 1$ for all $V \in I$. If this is the case, we call $K$ a splitting field for $G$.

We now present the isotypical decomposition in a more canonical form. Let $V$ be semi-simple. For each $U \in I$ we let $D(U)$ be its endomorphism algebra. Evaluation of endomorphisms makes $U$ into a left $D(U)$-module. The vector space $\text{Hom}_G(U, V)$ becomes a right $D(U)$-module via composition of
endomorphisms. The evaluation \( \text{Hom}_G(U, V) \otimes U \to V, \varphi \otimes u \mapsto \varphi(u) \) induces a linear map \( \iota_U : \text{Hom}_G(U, V) \otimes D(U) U \to V \).

\[ (1.10.8) \text{ Theorem.} \quad \text{Let} \quad \iota_U : \bigoplus_{U \in I} \text{Hom}_G(U, V) \otimes D(U) U \to V \quad \text{have components} \quad \iota_U. \quad \text{Then} \quad \iota \text{ is an isomorphism. The image of} \quad \iota_U \text{ is the} \ U \text{-isotypical component of} \ V. \]

\[ \text{Proof.} \quad \text{The maps} \quad \iota \text{ constitute, in the variable} \ V, \text{ a natural transformation on the category of semi-simple representations, and they are compatible with direct sums. Thus it suffices to consider irreducible} \ V. \quad \text{In that case, by Schur’s lemma, only the summand} \ \text{Hom}_G(V, V) \otimes D(V) V \quad \text{is non-zero, and evaluation is the canonical isomorphism} \ D(V) \otimes D(V) V \cong V. \quad \text{By construction,} \ \iota_U \quad \text{has an image in the} \ U \text{-isotypical part.} \]

Problems

1. Let \( U^* = \text{Hom}_K(U, K) \) be the dual vector space. This becomes a right \( G \)-representation via \( (g \cdot \varphi)(u) = \varphi(gu) \). The vector space \( \text{Hom}_G(U, KG) \) becomes a right representation via \( (\varphi \cdot g)(u) = \varphi(u) \cdot g \). Show: The linear map

\[ U^* \to \text{Hom}_G(U, KG), \quad \varphi \mapsto (u \mapsto \sum_{g \in G} \varphi(g^{-1}u)g) \]

is an isomorphism of right representations. An inverse morphism assigns to \( \alpha \in \text{Hom}_G(U, KG) \) the linear form \( U \to K \) which maps \( u \) to the coefficient of \( e \) in \( \alpha(u) \).

Let \( KG \) be semi-simple. Then the isotypical decomposition ?? of \( KG \) assumes the form

\[ \bigoplus_{U \in I} U^* \otimes D(U) U \to KG, \quad \varphi \otimes u \mapsto \sum_{g \in G} \varphi(g^{-1}u)g. \]

This is an isomorphism of left and right \( G \)-representations.

2. Use \[ (1.10.3) \] in order to show that we found (in section 1) enough irreducible complex representations of the dihedral group \( D_{2n} \). In the case that \( n \) is odd there are 2 one-dimensional and \( (n - 1)/2 \) two-dimensional irreducibles. In the case that \( n \) is even there are 4 one-dimensional and \( n/2 - 1 \) two-dimensional irreducibles.

3. Use \[ (1.10.7) \] in order to show that the representation of \( S_n \) on \( T_n = \{(x_i) \in K^n \mid \sum x_i = 0\} \) by permutation of coordinates is irreducible (\( K \) characteristic zero).

4. Let \( V \) be a \( KG \)-representation. Let \( V_G \) denote the subrepresentation spanned by the vectors \( v - gv, v \in V, g \in G \). Consider \( \alpha : V^G \to V/V_G \) induced by the inclusion \( V^G \subset V \). Show that \( \alpha \) is an isomorphism if the characteristic of \( K \) does not divide \( |G| \), and give an example where \( \alpha \) is not bijective.
Chapter 2

Characters

2.1 Characters

We assume in this chapter that $K$ has characteristic zero. It is then no essential restriction to assume moreover that $\mathbb{Q}$ is a subfield of $K$.

(2.1.1) Proposition. Let $U$ be a $G$-representation. Then the linear map

$$p: U \to U, \quad u \mapsto |G|^{-1} \sum_{g \in G} gu$$

is a $G$-equivariant projection onto the fixed point space $U^G$.

Proof. The map $p$ is the identity on $U^G$, equivariant by construction, and the image is contained in $U^G$. □

Let $V$ be a $G$-representation. We denote the trace of $l_g: V \to V$ by $\chi_V(g)$. The character of $V$ is the function $\chi_V: G \to K, g \mapsto \chi_V(g)$. The character of an irreducible representation is an irreducible character.

The trace of a projection operator is the dimension of its image. Therefore (2.1.1) yields the identity

$$\dim U^G = |G|^{-1} \sum_{g \in G} \chi_U(g).$$

(2.1)

Recall from linear algebra: The trace of a matrix is the sum of the diagonal elements, and conjugate matrices have the same trace. If we express $l_g$ in matrix form with respect to a basis, then the trace does not depend on the chosen basis. Since conjugate matrices have the same trace, isomorphic representations have the same character (1.1.1). Conjugation invariance also yields:

$$\chi_V(ghg^{-1}) = \chi_V(h), \quad g, h \in G.$$
Thus characters are class functions. From the matrix form of representations we derive some properties of characters:

\[
\chi_{V \oplus W} = \chi_V + \chi_W, \quad (2.2)
\]

\[
\chi_{V \otimes W} = \chi_V \chi_W, \quad (2.3)
\]

\[
\chi_V^*(g) = \chi_V(g^{-1}). \quad (2.4)
\]

Let \( V \) and \( W \) be \( G \)-representations. In section 1.8 we introduced the representation \( \text{Hom}_K(V, W) \) with fixed point set \( \text{Hom}_{KG}(V, W) \).

(2.1.2) Proposition. The character of \( \text{Hom}_K(V, W) \) is \( g \mapsto \chi_V(g^{-1}) \chi_W(g) \).

Proof. This is a consequence of \( [1.8.4, 2.3] \) and \( [2.4] \). We also prove it by a direct calculation with matrices, thus avoiding \( [1.8.4] \). We express the necessary data in matrix form. Let \( v_1, \ldots, v_m \) be a basis of \( V \) and \( w_1, \ldots, w_n \) a basis of \( W \). We set \( l^{-1}_g(v_i) = \sum_j a_{ij} v_j \) and \( l_g(w_k) = \sum_l b_{lk} w_l \). Then a basis of \( \text{Hom}_K(V, W) \) is \( e_{rs}: v_i \mapsto \delta_{si} w_r \). We compute:

\[
(g \cdot e_{rs})(v_i) = ge_{rs}(g^{-1} v_i) = ge_{rs}(\sum_j a_{ij} v_j) = g(\sum_j a_{ij} \delta_{sj} w_r) = \sum_{j,l} a_{sj} b_{lr} w_l = \sum s a_{ss} b_{rr} w_l = \sum s a_{ss} b_{rr} e_{li}(v_i).
\]

The trace is the sum of the diagonal elements \( \sum_{r,s} a_{ss} b_{rr} = \chi_V(g^{-1}) \chi_W(g) \).

We now combine 2.1.1 and 2.1.2 and obtain

\[
\langle V, W \rangle = \dim_K \text{Hom}_G(V, W) = |G|^{-1} \sum_{g \in G} \chi_V(g^{-1}) \chi_W(g). \quad (2.5)
\]

This formula tells us that we can compute \( \langle V, V \rangle \) from the character. The character does not change under field extensions. We know that \( \langle V, V \rangle = 1 \) implies that \( V \) is irreducible; it then remains irreducible under field extensions. If this is the case, we call the representation absolutely irreducible. When \( K \) is algebraically closed, Schur’s lemma says \( \langle V, V \rangle = 1 \). Therefore \( V \) is absolutely irreducible if and only if \( \langle V, V \rangle = 1 \).

(2.1.3) Theorem. Two representations of \( G \) are isomorphic if and only if they have the same character.

Proof. Let \( V \) and \( V' \) have the same character. Then, by \( [2.5] \), the values \( \langle W, V \rangle \) and \( \langle W, V' \rangle \) are equal for all \( W \). From \( [1.9.5] \) we now see that the multiplicities of \( W \in \text{Irr}(G, K) \) in \( V \) and \( V' \) are equal.
(2.1.4) **Theorem.** $G$-representations $V$ and $W$ are isomorphic if and only if $\text{res}_H^G V$ and $\text{res}_H^G W$ are isomorphic for each cyclic subgroup $H$ of $G$. 

(2.1.5) **Proposition.** Let $V = KS$ be the permutation representation of the finite $G$-set $S$. Then $\chi_V(g) = |S^g|$. Here $S^g = \{ s \in S \mid gs = s \}$.

**Proof.** Consider the matrix of $l_g$ with respect to the basis $S$. A basis element $s \in S$ yields a non-zero entry on the diagonal if and only if $gs = s$, and this entry is 1.

(2.1.6) **Proposition.** Let $K$ be a splitting field for $G$ and $H$. Then $\text{Irr}(G; K) \times \text{Irr}(H; K) \to \text{Irr}(G \times H; K)$, $(V, W) \mapsto V \otimes W$ is a well-defined bijection.

**Proof.** In the statement of the proposition we view $V \otimes W$ as $G \times H$-representation, as explained in section 1.8. From $\chi_{V \otimes W}(g, h) = \chi_V(g)\chi_W(h)$ and (2.5) we obtain

$$\langle V_1 \otimes W_1, V_2 \otimes W_2 \rangle_{G \times H} = \langle V_1, V_2 \rangle_G \langle W_1, W_2 \rangle_H.$$ 

This shows that $V \otimes W$ is irreducible, if we start with irreducible representations $V$ and $W$. It also shows that the map in question is injective. We use 1.10.3 and see that we got the right number of irreducible $G \times H$-representations. 

**Problems**

1. Let $H \triangleleft G$ and $V$ a $G$-representation. Then $V^H$ is a $G/H$-representation. Its character is given by $\chi_{V^n}(gH) = |H|^{-1} \sum_{h \in H} \chi_V(gh)$.

### 2.2 Orthogonality

We derive orthogonality properties of characters and show that the irreducible characters form an orthonormal basis in the ring of class functions. We assume that $K$ has characteristic zero and is a splitting field for $G$.

Let $Cl(G, K) = Cl(G)$ be the ring of class functions $G \to K$ (pointwise addition and multiplication). We define on $Cl(G)$ a symmetric bilinear form

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g^{-1})\beta(g).$$

(2.6)

Bilinearity is clear and the reason for symmetry is that we can replace summation over $g$ by summation over $g^{-1}$. By (2.5), $\langle V, W \rangle = \langle \chi_V, \chi_W \rangle$. This gives us together with Schur’s lemma the orthogonality properties of characters:
(2.2.1) Proposition. The irreducible characters form an orthonormal system with respect to the bilinear form \( 2.6 \).

The main result \( 2.2.5 \) of this section says that the irreducible characters are a basis of the vector space \( \text{Cl}(G) \). We prepare for the proof.

(2.2.2) Proposition. The linear map \( q_\alpha = \sum_{g \in G} \alpha(g)l_g : V \to V \) is a morphism for each representation \( V \) if and only if \( \alpha : G \to K \) is a class function.

Proof. Let \( \alpha \) be a class function. We compute
\[
q_\alpha(hv) = \sum \alpha(g)l_g(hv) = \sum \alpha(h^{-1}gh)h(h^{-1}gh) = hq_\alpha(v).
\]
For the converse we evaluate the equation \( q_\alpha(h) = hq_\alpha(e) \) in the regular representation and compare coefficients.

(2.2.3) Proposition. Let \( \alpha \) be a class function. Then \( p_\alpha = \sum_{g \in G} \alpha(g^{-1})l_g \) acts on \( V \) as the multiplication by the scalar \( |G|(\dim V)^{-1}\langle \alpha, \chi_V \rangle \).

Proof. By \( 2.2.2 \) \( p_\alpha \) is an endomorphism of \( V \), and \( \langle V, V \rangle = 1 \) tells us that \( p_\alpha \) is the multiplication with some scalar \( \lambda \). The computation \( (\text{Tr} = \text{Trace}) \)
\[
\lambda \dim V = \text{Tr}(\lambda \cdot \text{id}) = \text{Tr} \left( \sum \alpha(g^{-1})l_g \right)
= \sum \alpha(g^{-1}) \text{Tr}(l_g) = \sum \alpha(g^{-1})\chi_V(g) = |G|\langle \alpha, \chi_V \rangle.
\]
determines \( \lambda \).

(2.2.4) Lemma. Let \( \alpha \in \text{Cl}(G) \) be orthogonal to the characters of irreducible representations. Then \( \alpha = 0 \).

Proof. The hypothesis of the lemma and \( 2.2.3 \) imply that \( p_\alpha \) acts as zero morphism in each irreducible representation, hence in each representation. In the regular representation we have \( 0 = p_\alpha(e) = \sum g \alpha(g^{-1})g \). Hence \( \alpha(g) = 0 \) for all \( g \in G \).

(2.2.5) Theorem. The irreducible characters of \( G \) are an orthonormal basis of \( \text{Cl}(G) \). The number of irreducible representations is equal to the number of conjugacy classes of \( G \).

Proof. Let \( U \subset \text{Cl}(G) \) be a linear subspace. If \( U \neq \text{Cl}(G) \) then the orthogonal complement \( U^\perp \) with respect to \( \langle -, - \rangle \) is different from zero, since \( U^\perp \) is the kernel of the linear map \( \text{Cl}(G) \to \text{Hom}(U, K), x \mapsto \langle u \mapsto \langle x, u \rangle \rangle \). For the subspace \( U \) generated by characters, \( U^\perp = 0 \), by \( 2.2.4 \) hence \( U = \text{Cl}(G) \). Now recall \( 2.2.1 \).

The dimension of \( \text{Cl}(G) \) is the number of conjugacy classes, because a basis of \( \text{Cl}(G) \) consists of those functions which have value 1 on one class and value 0 on all the other classes.
There exist more general orthogonality relations. They are concerned with the entries of matrix representations and are consequences of the next result.

**(2.2.6) Proposition.** Let \( V \) be irreducible and suppose that \( \langle V, V \rangle = 1 \). Then for each linear map \( f \in \text{Hom}(V, V) \)

\[
\frac{1}{|G|} \sum_{g \in G} l_g f l_g^{-1} = \frac{\text{Tr}(f)}{\dim V} \cdot \text{id}.
\]

**Proof.** The left hand side is contained in \( \text{Hom}_G(V, V) \) and has the form \( \lambda \cdot \text{id} \), since \( \langle V, V \rangle = 1 \). We apply the trace operator

\[
\lambda \dim V = \text{Tr}(\lambda \cdot \text{id}) = |G|^{-1} \sum_{g \in G} \text{Tr}(l_g f l_g^{-1}) = |G|^{-1} \sum_{g \in G} \text{Tr}(f) = \text{Tr}(f)
\]

and determine \( \lambda \).

For \( v \in V \) and \( \varphi \in V^* \) we obtain from (2.2.6)

\[
\sum_{g \in G} \varphi(g f(g^{-1} v)) = (\dim V)^{-1} |G| \text{Tr}(f) \varphi(v).
\]

We apply this to the linear map \( f: v \mapsto \psi(v)w, \ w \in V, \ \psi \in V^* \) with trace \( \text{Tr}(f) = \psi(w) \) and obtain

\[
(G)^{-1} \sum_{g \in G} \varphi(g^{-1} v) \varphi(gw) = (\dim V)^{-1} \varphi(v) \psi(w).
\]

Note that we can use the definition (2.6) of \( \langle \alpha, \beta \rangle \) for arbitrary functions \( \alpha, \beta: G \to K \). This remark can be applied to the left hand side of (2.7).

Let \( v_1, \ldots, v_n \) be a basis of \( V \) and \( \varphi_1, \ldots, \varphi_n \) the dual basis. In a matrix representation \( g v_i = \sum_j r_{ji}^V(g) v_j \) we have \( \varphi_j(g v_i) = r_{ji}^V(g) \). We apply (2.7) to this situation and arrive at the following:

**(2.2.7) Orthogonality for matrix entries.** Let \( V \) and \( W \) be irreducible representations of \( G \). Then

\[
\langle r_{lk}^V, r_{ji}^W \rangle = \frac{1}{\dim V} \delta_{li} \delta_{jk} \delta_{VW}.
\]

We have treated the case \( V = W \). If \( V \) is not isomorphic to \( W \) and \( f \in \text{Hom}(V, W) \), then the left hand side of the equality in (2.8) is zero.

**Problems**

1. Let \( V_1, \ldots, V_r \) be a complete set of pairwise non-isomorphic irreducible \( KG \)-representations. Let \( (a_{ij}^r) \) denote a matrix representation of \( V_i \). Then the functions \( a_{ij}^r \) are an orthogonal basis of the space of functions \( G \to K \) with respect to the form (2.6).
2.3 Complex Representations

We begin with some special and useful properties of representations over the complex numbers.

An (Hermitian) inner product $V \times V \to \mathbb{C}$, $(u, v) \mapsto \langle u, v \rangle$ on a $G$-representation $V$ is called $G$-invariant if $\langle gu, gv \rangle = \langle u, v \rangle$ for $g \in G$ and $u, v \in V$. A representation together with a $G$-invariant inner product is a unitary representation. A real representation together with a $G$-invariant inner product is an orthogonal representation.

(2.3.1) Proposition. A complex representation $V$ of a finite group possesses a $G$-invariant inner product.

Proof. Let $b: V \times V \to \mathbb{C}$ be any inner product (conjugate-linear in the first variable) and define $c(u, v) = \frac{1}{|G|} \sum_{g \in G} b(gu, gv)$. Then $c$ is linear in $v$, conjugate linear in $u$, and $G$-invariant because of the averaging process. Also it is positive definite and $c(u, v) = \overline{c(v, u)}$. □

Let $U$ be a sub-representation of a unitary representation $V$. Then the orthogonal complement $U^\perp$ is again a sub-representation an $V = U \oplus U^\perp$. This gives another proof that complex representations are semi-simple. Similarly for orthogonal representations. If we choose an orthonormal basis in an $n$-dimensional unitary representation, then the associated matrix representation is a homomorphism into the unitary group $G \to U(n)$. In terms of matrix representations, (2.3.1) has the interesting consequence that a homomorphism $G \to \text{GL}_n(\mathbb{C})$ of a finite group $G$ is conjugate to a homomorphism $G \to U(n)$.

Let $V$ be a complex representation. There is associated the complex-conjugate representation $G \times \overline{V} \to \overline{V}$ on the conjugate vector space $\overline{V}$ (the same underlying set and vector addition, but $\lambda \in \mathbb{C}$ now acts as multiplication with $\overline{\lambda}$).

We consider a Hermitian form on $V$ as a bilinear map $\overline{V} \times V \to \mathbb{C}$. Associated is the adjoint $\overline{V} \to V^*$, $v \mapsto (u \mapsto \langle v, u \rangle)$ into the dual vector space. It is an isomorphism of $G$-representations, in the case of a $G$-invariant inner product. In terms of characters this means $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. For complex class functions we define a Hermitian form on $\text{Cl}(G)$ by

\[(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g). \quad (2.9)\]

The relation $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ shows $(\chi_V, \chi_W) = \langle V, W \rangle$. Therefore the irreducible characters are also an orthonormal basis for this form. Recall the notation $I = \text{Irr}(G; \mathbb{C})$. 

Let $C \subset G$ be a representing system for the conjugacy classes. From $|C| = |I|$ we see that $X: C \times I \to K$, $(c, V) \mapsto \chi_V(c)$ is a square matrix. It is called the character table of $G$.

We express the orthogonality relations in terms of the character table. Let $X^*: I \times C \to \mathbb{C}$, $(V, c) \mapsto \overline{\chi_V(c)}$ be the conjugate-transpose and $D: C \times C \to \mathbb{C}$ the diagonal matrix $(c, d) \mapsto \delta_{c,d}|c|$, where $|c|$ denotes the cardinality of the conjugacy class of $c$. The orthogonality relation $(\chi_V, \chi_W) = \delta_{V,W}$ then reads:

**2.3.2 First orthogonality relation.** For irreducible complex representations $V$ and $W$ the relation

$$\sum_{c \in C} |c| \overline{\chi_V(c)} \chi_W(c) = |G| \delta_{V,W}$$

holds. \hfill \Box

In matrix form 2.3.2 says $X^*DX = |G|E$ (unit matrix $E$). This implies

$$XX^*D = XX^*DXX^{-1} = X(|G|E)X^{-1} = |G|E$$

and then $XX^* = |G|D^{-1}$. Let $Z(c) = \{g \in G \mid gcg^{-1} = c\}$ denote the centralizer of $c$ in $G$. Then $|c| = |G/Z(c)|$. We write out the last matrix equation:

**2.3.3 Second orthogonality relation.** For $c, d \in C$ the relation

$$\sum_{V \in I} \overline{\chi_V(c)} \chi_V(d) = \delta_{c,d}|Z(c)|$$

holds. \hfill \Box

**2.3.4 Proposition.** Let $(V, \langle -,- \rangle_V)$ and $(W, \langle -,- \rangle_W)$ be unitary representations. Suppose $V$ and $W$ are isomorphic as complex representations. Then they are isomorphic as unitary representations, i.e., there exists a $G$-morphism $f: V \to W$ such that $\langle f(v_1), f(v_2) \rangle_W = \langle v_1, v_2 \rangle_V$.

*Proof.* Let $\varphi: V \to W$ be a $G$-morphism. We use $\varphi$ to pull $\langle -,- \rangle_W$ back to $V$, i.e., we define a second inner product $\langle -,- \rangle'$ on $V$ by $\langle v_1, v_2 \rangle' = \langle \varphi(v_1), \varphi(v_2) \rangle_W$. It suffices to produce a $G$-morphism $\gamma: V \to V$ such that $\langle \gamma(v_1), \gamma(v_2) \rangle = \langle v_1, v_2 \rangle'$. We choose an orthonormal basis $B$ of $V$ with respect to $\langle -,- \rangle$ and express everything with respect to this basis. Then $\langle -,- \rangle'$ becomes the standard inner product. There exists a positive definite Hermitian matrix $A$ such that $\langle u, v \rangle' = \langle u, Av \rangle = u'Av$. Since $\langle -,- \rangle'$ is $G$-invariant, $l_gA = Al_g$. Let $C = \sqrt{A}$ be a positive definite Hermitian matrix. The matrix $C$ also commutes with $l_g$, since it is a limit of polynomials in $A$. Then $C$ defines a morphism $\gamma$ with the desired properties. \hfill \Box
(2.3.5) Corollary. Let $\alpha, \beta : G \to U(n)$ be unitary representations which are conjugate in $GL_n(\mathbb{C})$. Then they are conjugate in $U(n)$. The conjugation matrix can be chosen in $SU(n)$, i.e., to have determinant one. □

We list a few more properties of complex characters.

(2.3.6) Proposition. Let $\chi$ be the character of a complex representation $V$. Then:

1. $\chi(1) = \dim V$.
2. $|\chi(g)| \leq \chi(1)$.
3. $|\chi(g)| = \chi(1)$ if and only if $l_g$ is the multiplication by a scalar.
4. $\chi(g) = \chi(1)$ if and only if $g$ is contained in the kernel of $V$.

Proof. (1) The trace of the identity is $\dim V$.

(2) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $l_g$. They are roots of unity, and $\chi(g) = \lambda_1 + \cdots + \lambda_n$. Hence $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \chi(1)$.

(3) If equality holds, then $\lambda_1 = \cdots = \lambda_n = \lambda$ and $l_g$ is multiplication by $\lambda$.

(4) By (3), $l_g$ is the multiplication by 1, if $\chi(g) = \chi(1)$. □

(2.3.7) Remark. If $G$ has a normal subgroup different from 1 and $G$, then there exists a nontrivial irreducible character $\chi$ and $1 \neq g \in G$ such that $\chi(g) = \chi(1)$. Conversely, from ?? we see, that if $\chi$ and $g$ with these properties exist, then $G$ has a nontrivial normal subgroup. We see that one can obtain group theoretic information from the character table. ◊

The values of complex characters are very special complex numbers. The value $\chi_V(g)$ is the sum of the eigenvalues of $l_g$, and these eigenvalues are $|g|$-roots of unity ($|g|$ order of $g$). Let $\mathbb{Z}[\zeta]$ be the subring of the field $\mathbb{Q}(\zeta)$ generated by $\zeta$. The exponent of a group is the least common multiple of the orders of its elements. Let $\zeta$ be a primitive $n$-root of unity, say $\zeta = \exp(2\pi i/n)$, $n$ the exponent of $G$. Then $\chi_V$ has values in $\mathbb{Z}[\zeta]$. In number theory, the ring $\mathbb{Z}[\zeta]$ is the ring of algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta)$. (An algebraic integer is the root of a monic polynomial with coefficients in $\mathbb{Z}$.)

Problems

1. Express the orthogonality relations ?? for complex representations using the Hermitian form (2.9).
2. Let $V, W$ be orthogonal representations of $G$ which are isomorphic as real representations. Then they are isomorphic as orthogonal representations.
3. Let $V, W$ be real representations. If they are isomorphic, considered as complex representations, then they are isomorphic as real representations. What does this imply for matrix representations?
4. The character table is a square matrix. Determine the absolute value of its determinant.
2.4 Examples

We study in some detail the groups $A_4, S_4, A_5$. The symmetric group $S_n$ is the permutation group of $[n] = \{1, \ldots, n\}$. The alternating group is the normal subgroup of $S_n$ of even permutations. The geometric significance of the groups in question comes from Euclidean geometry: $A_4, S_4, A_5$ are the symmetry groups of the tetrahedron, octahedron (cube), icosahedron (dodecahedron), respectively (as far as rotations are concerned, i.e., as subgroups of $SO(3)$).

We begin with some general remarks about permutations. Let $\pi \in S_n$. The cyclic group generated by $\pi$ acts on $[n]$. We decompose $[n]$ into orbits under this action. An orbit has the form

$$(x, \pi(x), \pi^2(x), \ldots, \pi^{t-1}(x)), \quad \pi^t(x) = x$$

where $t$ is the length of the orbit. We call an orbit a cycle of the permutation. A permutation can be recovered from its cycles. Therefore we use the cycles to denote the permutation. As an example, the permutation $(318496527) \in S_9$ has the cycles

$$(1, 3, 8, 2), (4), (5, 9, 7).$$

This means, e.g., that $5 \mapsto 9, 9 \mapsto 7, 7 \mapsto 5$. A cyclic permutation of the entries in a cycle does not change its meaning; thus $(5, 9, 7) = (9, 7, 5) = (7, 5, 9)$. In practice it is not necessary to write cycles of length one, since they just describe fixed points of the permutations. The conceptual significance of the cycles is:

(2.4.1) Proposition. Permutations in $S_n$ are conjugate elements of the group if and only if for each $k \in \mathbb{N}$ they have the same number of cycles of length $k$. □

A partition of $n$ is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 1$ and $\sum_{j=1}^r \lambda_j = n$. A conjugacy class of $S_n$ is determined by its associated partition; the $\lambda_j$ are the lengths of the cycles in the permutation. Thus we have found a combinatorial method to determine the number of irreducible complex representations of $S_n$; it is the number of partitions of $n$. The partitions $3, 21, 111$ of 3 tell us that $S_3$ has 3 irreducible representations.

(2.4.2) Proposition. Suppose $\pi \in S_n$ has $k(j)$ cycles of length $j$. The the automorphism group of $[n]$ has order $1^{k(1)} \cdot k(1)! \cdot 2^{k(2)} \cdot k(2)! \cdots n^{k(n)} \cdot k(n)!$. This is the order of the centralizer; hence $n!$, divided by this number, is the size of the conjugacy class of $\pi$. □

(2.4.3) Representations of $S_4$. There exist 5 partitions $1111, 211, 22, 31, 4$. We list representing elements of the conjugacy classes and the cardinality of
the conjugacy class in the next table. The second row can be obtained from 2.4.2.

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<thead>
<tr>
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<th>(12)</th>
<th>(12)(34)</th>
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<tbody>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>3</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

We turn to the determination of the character table and to the construction of irreducible representations. Names for the five representations and their characters are \( V_j \), \( 1 \leq j \leq 5 \). We already know 2 one-dimensional representations, the trivial representation \( V_1 \) and the sign-representation \( V_2 \). Their characters are easily computed.

<table>
<thead>
<tr>
<th>Character table of ( S_4 )</th>
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<tbody>
<tr>
<td>( V_1 )</td>
</tr>
<tr>
<td>( V_2 )</td>
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<tr>
<td>( V_3 )</td>
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<tr>
<td>( V_4 )</td>
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<tr>
<td>( V_5 )</td>
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</tbody>
</table>

We also know already a three-dimensional representation on the space \( V_4 = \{(x_1, x_2, x_3, x_4) | \sum x_i = 0\} \) by permutation of coordinates. We check again that it is irreducible by computing its character. The character of the permutation representation on \( C^4 \) is easily determined by 2.1.5 to have the values \( 4, 2, 0, 1, 0 \). We have to subtract the character of the trivial representation; the result is given in the table. The computation

\[
\langle V_4, V_4 \rangle = \frac{1}{24T} \sum g |\chi_{V_4}(g)|^2 = 3^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot (-1)^2 = 1
\]

shows that \( V_4 \) is irreducible. The character of \( V_5 = V_4 \otimes V_2 \) is seen to be as in the table. Thus we found another irreducible representation. We know that the remaining representation must be two-dimensional 1.10.3. It turns out that \( S_4 \) has a quotient \( S_3 \), the kernel contains \( (12)(34) \). We can lift a two-dimensional representation of \( S_3 \) to \( S_4 \). We lift the analogue of \( V_4 \) for \( S_3 \). (2.4.4)

(2.4.4) **Representations of \( A_5 \).** We begin again with the determination of the conjugacy classes. We use the cycle notation and have to start with even permutations. But now it is only allowed to conjugate with even permutations, and this has the effect that some of the conjugacy classes of \( S_5 \) can split in \( A_5 \) into two classes.

(2.4.5) **Proposition.** Let \( e \in A_n \). Then the \( S_n \)-conjugacy class of \( e \) is contained in \( A_n \). The \( S_n \)-conjugacy class of \( e \) split into two \( A_n \)-conjugacy classes if and only if the centralizers of \( e \) in \( A_n \) and \( S_n \) coincide. This happens if and only if the partition associated to \( e \) consists of different odd numbers. \( \diamond \)
Let $\zeta$ be a primitive 5-th root of unity; then $-\alpha = \zeta + \zeta^{-1}$ and $-\beta = \zeta^2 + \zeta^{-2}$. The representation $V_1$ is trivial. $V_4$ is the permutation representation on $\{(x_1, \ldots, x_5) | \sum x_j = 0\}$. The group $A_5$ has a subgroup $H \cong D_{10}$. The representation $V_5$ is obtained from the permutation representation $C(A_5/H)$ by subtracting the trivial representation. The remaining two representations must be three-dimensional, since $60 - 1^2 - 4^2 - 5^2 = 18 = 3^2 + 3^2$. It is possible to determine the characters without construction of the representations; one uses the fact that the representations cannot have a kernel; and that $z^j$ and $z^{-j}$ are conjugate, so that the character values are real and sums of 5-th roots of unity. The group $A_5$ has an outer automorphism which interchanges $z = (12345)$ and $z^2 = (13524)$; it is obtained by conjugation with $(2354)$; one verifies $(2354) \circ z \circ (4532) = z^2$. The representation $V_3$ is obtained from $V_2$ by this automorphism; and $V_2$ has a realization over $\mathbb{R}$ as orthogonal symmetry group of the icosahedron.

Problems

1. $S_4$ acts by conjugation on the set of even permutations of order two. Show that this induces a surjection $S_4 \to S_3$.
2. Compute the number of elements in $A_5$ of a given order.
3. Determine the irreducible representations and the character table for $A_4$.
4. Decompose the tensor product of irreducible representations for $G = A_4, S_4, A_5$.
5. Show that $S_5$ has seven conjugacy classes and irreducible complex representations of dimensions 1, 1, 4, 4, 5, 5, 6.
6. For a partition $(\lambda_1, \ldots, \lambda_r)$ of $n$ let $S(\lambda) = S(\lambda_1) \times \cdots \times S(\lambda_r)$. Set $V(\lambda) = C(S_n/S(\lambda))$. Decompose these permutation representations in the cases $S_4$ and $S_5$ into irreducibles.


2.5 Real and Complex Representations

Let $W$ be a $KG$-representation. The involution $T: W \otimes W \to W \otimes W, x \otimes y \mapsto y \otimes x$ is a morphism of representations. If $K$ has characteristic different from 2, we split $W \otimes W$ into the $\pm 1$-eigenspaces,

$$S^2(W) = (W \otimes W)_+,$$
$$\Lambda^2(W) = (W \otimes W)_-. $$

$S^2(W)$ is the second symmetric power of $W$ and $\Lambda^2(W)$ the second exterior power. Suppose now that $K = \mathbb{C}$. If $w_1, \ldots, w_n$ is a basis of $W$, then $g_{ij} = \frac{1}{2}(w_i \otimes w_j + w_j \otimes w_i), i \leq j$ is a basis of $S^2(W)$ and $w_{ij} = \frac{1}{2}(w_i \otimes w_j - w_j \otimes w_i), i < j$ is a basis of $\Lambda^2(W)$. From this information we compute the characters

$$\chi_{W \otimes W}(g) = \chi_W(g)^2$$
$$\chi_{S^2W}(g) = \frac{1}{2}(\chi_W(g)^2 + \chi_W(g^2))$$
$$\chi_{\Lambda^2W}(g) = \frac{1}{2}(\chi_W(g)^2 - \chi_W(g^2)).$$

Let $W$ be irreducible. Then $W^* \cong \overline{W}$ is irreducible too. Therefore $\langle W \otimes W, 1 \rangle = \langle W, W^* \rangle$ is 1 if $W \cong W^*$ and 0 otherwise. An isomorphism $W \cong \overline{W}$ exists if and only if $\chi_W$ is real-valued. In that case we call $W$ self-conjugate. Elements in $\text{Hom}_G(S^2W, \mathbb{C})$ are symmetric $G$-invariant bilinear forms, and elements in $\text{Hom}_G(\Lambda^2W, \mathbb{C})$ are skew-symmetric $G$-invariant bilinear forms. Hence $W$ carries a non-zero $G$-invariant bilinear form if and only if $W$ is self-conjugate. Suppose $W$ is self-conjugate. From $\langle W \otimes W, 1 \rangle = 1, \langle S^2W, 1 \rangle = 1, \langle \Lambda^2W, 1 \rangle = 1$ we see that there are two cases: Either $\langle S^2W, 1 \rangle = 1, \langle \Lambda^2W, 1 \rangle = 0$ or $\langle S^2W, 1 \rangle = 0, \langle \Lambda^2W, 1 \rangle = 1$.

\begin{enumerate}
\item \textbf{Proposition.} Let $W$ be irreducible. Then

$$\sigma(W) = \frac{1}{|G|} \sum_{g \in G} \chi_W(g^2) = \begin{cases} 0 & W \not\cong W^* \\ 1 & \langle S^2W, 1 \rangle = 1 \\ -1 & \langle \Lambda^2W, 1 \rangle = 1 \end{cases}$$

\item Suppose $W$ is the complexification of a real representation $U$, i.e., $W \cong \mathbb{C} \otimes U = U_\mathbb{C}$. Then $U$ is irreducible and carries a symmetric $G$-invariant $\mathbb{R}$-bilinear form. This form extends to a symmetric $G$-invariant $\mathbb{C}$-bilinear form on $W$. Hence in this case $\sigma(W) = 1$. The converse is also true.

\item \textbf{Proposition.} Suppose $W \in \text{Irr}(G, \mathbb{C})$ carries a $G$-invariant symmetric form. Then $W$ is the complexification of a real representation.
\end{enumerate}
Chapter 3

The Group Algebra

3.1 The Theorem of Wedderburn

Assume that the characteristic of $K$ does not divide the order of $G$. Then the group algebra $A = KG$, considered as a left $KG$-representation, is semi-simple and hence a direct sum of its isotypical parts. Recall that sub-representations of $KG$ are the same thing as left ideals or as submodules.

(3.1.1) Proposition. Let $A(V)$ denote the $V$-isotypical part of $A$ belonging to $V \in \text{Irr}(G; K)$. The $A(V)$ is a two-sided ideal, and every two-sided ideal is a direct sum of isotypical parts.

Proof. Let $V \subset A$ be irreducible and $a \in A$. Then $W = Va$ is a left ideal, and by Schur’s lemma $r_a : V \rightarrow W, x \mapsto xa$ is either zero or an isomorphism. Hence $W \subset A(V)$.

Let $V, W$ be isomorphic irreducible left ideals. Since $A$ is semi-simple, there exists a projection $f : A \rightarrow V$. Let $s : V \rightarrow W$ be an isomorphism. Then $fs(x) = fs(x \cdot 1) = x \cdot sf(1) = xa$, $a = sf(1)$. If $x \in V$, then $f(x) = x$ and hence $sf(x) = s(x) = xa$, i.e., $W = Va$.

Let $B \subset A$ be a two-sided ideal. Let $V \subset B$ be irreducible and also $W \subset A(V)$. Then $W = Va$. Since $B$ is a right ideal, $W \subset B$ and hence $A(V) \subset B$.

The isotypical parts are therefore the minimal two-sided ideal. A two-sided ideal is itself an associative algebra, with addition and multiplication inherited from $A$.

(3.1.2) Proposition. Let $A = A_1 \oplus \cdots \oplus A_r$ be the decomposition into the minimal two-sided ideal. Then $A_i A_j = 0$ for $i \neq j$. 
Proof. Let $I \subset A_i$ and $J \subset A_j$ be irreducible left ideals. Then $A_i \supset IJ \subset A_j$, since $A_i, A_j$ are two-sided. From $A_i \cap A_j = 0$ we see $IJ = 0$.

Let $1 = e_1 + \cdots + e_r$, $e_j \in A_j$ be the decomposition of the unit element. Then $e_i^2 = e_i$ and $e_ie_j = 0$ for $i \neq j$. This follows from $\sum e_j = 1 = 1 \cdot 1 = (\sum e_j)(\sum e_k) = \sum e_je_k = \sum e_j^2$.

A consequence: $e_j$ is the unit element of the algebra $A_j$.

Let $B$ be a minimal two-sided ideal of $A$. A linear subspace $V$ of $B$ is a $B$-submodule if and only if it is an $A$-submodule. The map $r: B \to \text{Hom}_B(B, B)$, $x \mapsto rx$ is because of $rxry = r_{yx}$ an anti-isomorphism of algebras. Let $B \cong V^n$, where $V$ is an irreducible submodule of $B$. Then $\text{Hom}_B(B, B) \cong \text{Hom}_B(V^n, V^n)$. The latter is, by the rules of linear algebra, a matrix algebra. Let $i_j: V \to V^n$ be the inclusion of the $j$-the summand and $p_k: V^n \to V$ be the projection onto the $k$-summand. We associate to $f \in \text{Hom}_B(V^n, V^n)$ the matrix $(f_{jk})$, $f_{jk} = p_k fi_j \in \text{End}_B(V) = D$. Since $V$ is irreducible, $D$ is a division algebra. Therefore $\text{Hom}_B(V^n, V^n)$ is isomorphic to the matrix algebra $M_n(D)$ of $(n, n)$-matrices with entries in $D$. Passage to the transposed matrix is an isomorphism $M_n(D^\circ) \cong M_n(D)$. (Notation: $C^\circ$ the algebra opposite to $C$, i.e., order of the multiplication interchanged.) Therefore we have shown in our context:

(3.1.3) Theorem (Theorem of Wedderburn). The minimal left ideals of the group algebra are isomorphic to matrix-algebras $M_n(D^\circ)$; here $D = \text{End}(V)$ if $B$ is the $V$-isotypical part, and $n$ is the multiplicity of $V$ in $B$.

If $K = C$, then the division algebras appearing are just the field $C$ itself. In the next section we describe the decomposition into matrix algebras in a more explicit manner and relate it to character theory.

3.2 The Structure of the Group Algebra

We assume in this section that $K$ is a splitting field for $G$ of characteristic zero. We write $\dim V = |V|$.

(3.2.1) Proposition. Let $V \in \text{Irr}(G; K)$. The assignment $t_V: \text{Hom}(V, V) \to KG, \ \alpha \mapsto \frac{|V|}{|G|} \sum_{g \in G} \text{Tr}(l^{-1}_g \alpha)g$ is a homomorphism of algebras.
Proof. By comparing coefficients in $K G$ we see that the statement amounts to
\[ \frac{|V|}{|G|} \sum_{g \in G} \text{Tr}(l_g^{-1} \alpha) \text{Tr}(l_x^{-1} l_g \beta) = \text{Tr}(l_x^{-1} \alpha \beta) \]
for $\alpha, \beta \in \text{Hom}(V, V)$ and $x \in G$. It suffices to prove this for $x = e$.

We use the relation 2.2.6
\[ \frac{|V|}{|G|} \sum_{g \in G} l_g \beta l_g^{-1} = \text{Tr}(\beta) \text{id}_V, \quad \beta \in \text{Hom}(V, V) \quad (3.1) \]
and compute
\[ \sigma = \frac{|V|^2}{|G|^2} \sum_{g, u \in G} l_g \beta l_u l_g^{-1} l_u^{-1} \]
in two ways. We apply (3.1) to $\sum_{u} l_u l_g^{-1} l_u^{-1}$ and use the definition of $\chi_V$; this shows us that $\sigma$ is equal to
\[ \frac{|V|}{|G|} \sum_{g} \chi_V(g^{-1}) l_g \beta \alpha. \]
The endomorphism $\frac{|V|}{|G|} \sum_{g} \chi_V(g^{-1}) l_g$ is the identity on $V$. Hence $\sigma = \beta \alpha$. We now apply (3.1) to $\sum_{u} l_u l_g^{-1} l_u^{-1}$ and obtain
\[ \sigma = \frac{|V|}{|G|} \sum_{u} \text{Tr}(\beta l_u) l_u^{-1} \alpha. \]
Finally we apply the trace operator to this equation and arrive at
\[ \text{Tr}(\beta \alpha) = \frac{|V|}{|G|} \sum_{u} \text{Tr}(l_u \beta) \text{Tr}(l_u^{-1} \alpha), \]
and this was to be shown. \hfill \qed

The homomorphism $t_V$ is moreover a morphism of $(G, G)$-representations, i.e., one verifies directly from the definitions that $t_V(l_g \alpha l_h) = g t_V(\alpha) h$.

**3.2.2 Proposition.** The image of $t_V$ is the $V$-isotypical part of $K G$. The $(G, G)$-representation $\text{Hom}(V, V)$ is irreducible and the image of $t_V$ is the $\text{Hom}(V, V)$-isotypical part of $K G$ as a $(G, G)$-representation.

**Proof.** The canonical map $V^* \otimes V \to \text{Hom}(V, V)$ is an isomorphism of representations. By 2.1.6 these representations are irreducible. Since $t_V$ is non-zero, $t_V$ is injective. Certainly $t_V$ has an image in the $V$-isotypical part. We know already that it has dimension $|V|^2 = \dim \text{Hom}(V, V)$. Therefore $t_V$ maps isomorphically onto the $V$-isotypical part. \hfill \qed

The homomorphisms $t_V$ combine to an isomorphism of algebras
\[ t : \bigoplus_{V \in I} \text{Hom}(V, V) \to K G, \quad (x_V) \mapsto \sum_{V \in I} t_V(x_V). \]
This isomorphism induces an isomorphism of the centers of the algebras. The center of $\text{Hom}(V, V)$ consists of the multiples of the identity. Let $Z(A)$ denote the center of the algebra $A$. We obtain a homomorphism of algebras
\[ \tau_V = \text{pr}_V \circ t^{-1} : Z(KG) \to Z(\text{Hom}(V, V)) \cong \{ \lambda \cdot \text{id} \mid \lambda \in K \} \cong K. \]
(3.2.3) Proposition. \( \tau_V(\sum_{g \in G} \alpha(g)g) = |V|^{-1} \sum_{g \in G} \alpha(g)\chi_V(g) . \)

Proof. The elements \( t_V(\text{id}_V) = e_V \) are, by 3.2.2, a vector space basis of \( Z(KG) \). Hence it suffices to verify the assertion for these elements. The verification amounts to 
\[
|W|^{-1} |G|^{-1} \sum_g \chi_W(g^{-1})\chi_V(g) = \delta_{VW},
\]
and this we know by 2.2.6. \( \square \)

(3.2.4) Proposition. The element \( \sum_{g \in G} \alpha(g)g \in KG \) is contained in the center of \( KG \) if and only if \( \alpha \) is a class function. \( \square \)

An element \( e \in A \) in an algebra \( A \) is called idempotent, if it satisfies \( e^2 = e \). Idempotents \( e, f \) are orthogonal, if \( ef = fe = 0 \). The elements \( e_V \in KG, V \in I \) are pairwise orthogonal, central idempotents. A central idempotent is called primitive if it is not the sum of two orthogonal (non-zero) idempotents. Since the \( e_V \) form a basis of the center of \( KG \), it is easy to verify that the \( e_V \) are primitive.

(3.2.5) Proposition. The multiplication by \( e_V \) is in each representation the projection onto the \( V \)-isotypical part. \( \square \)

(3.2.6) Proposition. Suppose \( V, W \in I(G; \mathbb{C}) \). Then the orthogonality relation 
\[
\sum_{g \in G} \chi_V(g^{-1})\chi_W(xg) = \frac{|W|}{|G|} \langle V, W \rangle \chi_V(x) \quad \text{holds.}
\]

Proof. The relation \( e_V e_W = (V, W)e_V \) says
\[
\frac{|V||W|}{|G|^2} \sum_{g, h} \chi_V(g^{-1})\chi_W(h^{-1})gh = \frac{|W|}{|G|} \langle V, W \rangle \sum_{x \in G} \chi_V(x^{-1})x.
\]

Now we compare the coefficients of \( x^{-1} \). \( \square \)
Chapter 4

Induced Representations

4.1 Basic Definitions and Properties

We compare representations of different groups. The ground field $K$ is fixed.

Let $H$ be a subgroup of $G$ and $V$ an $H$-representation. Recall the construction $X \times_H Y$ for a right $H$-set $X$ and a left $H$-set $Y$; it is the quotient of the product $X \times Y$ under the equivalence relation $(x,y) \sim (xh^{-1},hy)$. We denote equivalence classes by their representatives in $X \times Y$. We apply this construction to the right cosets $gH$, considered as $H$-sets by right multiplication. We use the bijection $i_g: V \to gH \times_H V, v \mapsto (g,v)$ to transport the vector space structure from $V$ to $gH \times_H V$. If we choose another representative $gh \in gH$ of the coset $gH$, then $i_{gh} = i_{gh}$, and therefore the vector space structure is well-defined. (Although this vector space is just a model of $V$, we want this model to depend on the coset.) We define a $G$-action on $\bigoplus_{gH \in G/H} gH \times_H V$; the element $u \in G$ acts as follows

$$gH \times_H V \to u gH \times_H V, \quad (g,v) \mapsto (ug,v).$$

We see that $G$ permutes the summands $gH \times_H V$ transitively. The resulting $G$-representation is called the induced representation, and is denoted by

$$\text{ind}_H^G V = \bigoplus_{gH \in G/H} gH \times_H V. \quad (4.1)$$

### Example (4.1.1)

Suppose $|G/H| = 2$. Let $h \mapsto A(h)$ be a matrix representation of $H$. Fix an element $g \in G \setminus H$. Then a matrix representation for $\text{ind}_H^G$ is

$$h \mapsto \begin{pmatrix} A(h) & 0 \\ 0 & A(g^{-1}bg) \end{pmatrix}, \quad gh \mapsto \begin{pmatrix} 0 & A(gbg) \\ A(h) & 0 \end{pmatrix}.$$ 

Verify this, using the bijections $i_e$ and $i_g$. \hfill \Diamond
There exist other constructions of the induced representation, from the view point of set-theory or algebra. Therefore we characterize it by a universal property. The bijection \( i_e : V \to H \times H, v \mapsto (e, v) \) preserves the \( H \)-action. We thus obtain an \( H \)-morphism

\[
i^G_H : V \to \text{ind}^G_H V.
\]

If \( W \) is a \( G \)-representation, we denote by \( \text{res}^G_H W \) the \( H \)-representation obtained from \( W \) by restricting the group action to \( H \). The universal property is:

**Proposition.** The assignment

\[
\text{Hom}_G(\text{ind}^G_H V, W) \to \text{Hom}_H(V, \text{res}^G_H W), \quad \Phi \mapsto \Phi \circ i^G_H
\]

is a natural isomorphism of vector spaces. In terms of dimensions this implies

\[
\langle \text{ind}^G_H V, W \rangle_G = \langle V, \text{res}^G_H W \rangle_H.
\]

**Proof.** From the construction of \( \text{ind}^G_H V \) we see that a \( G \)-morphism from \( \text{ind}^G_H V \) is determined by its restriction to the summand \( H \times H V \). Therefore the map in question is injective.

Conversely, given \( \varphi : V \to \text{res}^G_H W \), we define a \( G \)-morphism \( \Phi : \text{ind}^G_H V \to W \) on the summand \( gH \times_H V \) by \( (g, v) \mapsto g \cdot \varphi(v) \). Another representative \( (gh^{-1}, hv) \) leads to the same value \( gh^{-1} \varphi(hv) \), because \( \varphi \) is an \( H \)-morphism. Therefore \( \Phi \) is well-defined, a \( G \)-morphism by construction, and \( \Phi \circ i^G_H = \varphi \).

We refer to (4.1.2) as *Frobenius reciprocity*. Suppose \( j^G_H : V \to V \) is a \( H \)-morphism into a \( G \)-representation \( V \) such that \( \Phi \mapsto \Phi \circ j^G_H \) induces a bijection \( \text{Hom}_G(V, W) \cong \text{Hom}_H(V, \text{res}^G_H W) \). Then there exists a unique isomorphism \( \gamma : \text{ind}^G_H V \to V \) of \( G \)-representations such that \( j^G_H = \gamma \circ i^G_H \). This expresses the fact, that \( i^G_H \) determines the induced representation. One consequence of this fact is the transitivity of induction:

**Proposition.** Let \( A \subset B \subset C \) be groups. Then there exists a canonical \( C \)-isomorphism \( \text{ind}^G_H \text{ind}^B_A V \cong \text{ind}^G_A V \) for \( A \)-representations \( V \), since \( i^G_H i^B_A \) has the universal property.

Given a \( G \)-representation, we often ask whether it can be induced from a subgroup \( H \). From the construction of \( \text{ind}^G_H V \) we obtain the following answer.

**Proposition.** Let \( V \) be an \( H \)-sub-representation of the \( G \)-representation \( W \). The subspace \( gV \subset W \) depends only on the coset \( gH \). We denote it therefore by \( gHV \). Suppose \( W \) is the direct sum of the subspaces \( gHV \). Then the canonical map \( \text{ind}^G_H V \to W \) associated by (4.1.3) to the inclusion \( V \subset W \) is an isomorphism.
An $H$-morphism $\alpha: V_1 \to V_2$ induces a $G$-morphism

\[ \text{ind}^G_H \alpha: \text{ind}^G_H V_1 \to \text{ind}^G_H V_2, \quad (g, v) \mapsto (g, \alpha(v)). \]

In this manner $\text{ind}^G_H$ becomes a functor from the category of $KH$-representations to the category of $KG$-representations. The isomorphism 4.1.3 is compatible with induced morphisms in the variables $V$ and $W$. In category theory one says that the induction functor $\text{ind}^G_H$ is left adjoint to the restriction functor $\text{res}^G_H$.

Induction preserves direct sums; we have a natural isomorphism

\[ \text{ind}^G_H(V_1 \oplus V_2) \cong \text{ind}^G_H V_1 \oplus \text{ind}^G_H V_2. \tag{4.2} \]

Let $W$ be a $G$-representation. Then the bijections

\[ gH \times_H (V \otimes \text{res}^G_H W) \to (gH \times_H V) \otimes W; \quad (g, v \otimes w) \mapsto (g, v) \otimes gw \]

combine to a natural isomorphism of $G$-representations

\[ \text{ind}^G_H(V \otimes \text{res}^G_H W) \cong (\text{ind}^G_H V) \otimes W. \tag{4.3} \]

(4.1.5) Example. Let $1_H$ denote the trivial one-dimensional $H$-representation. Then $\text{ind}^G_H 1_H$ is the permutation representation $K(G/H)$. The basis element $gH \in K(G/H)$ corresponds to $(g, 1) \in gH \times_H K$.

If $V$ happens to be the restriction of a $G$-representation $V = \text{res}^G_H W$, then

\[ \text{ind}^G_H(V) \cong \text{ind}^G_H(1_H \otimes V) \cong (\text{ind}^G_H 1_H) \otimes W \cong K(G/H) \otimes W. \]

In general one can think of $\text{ind}^G_H V$ as a kind of mixture of the permutation representation $K(G/H)$ with $V$.

We compute the character of an induced representation in the case that $K$ has characteristic zero.

(4.1.6) Proposition. Let $W = \text{ind}^G_H V$. Then the character of $W$ is given by the formula

\[ \chi_W(u) = \sum_{gH \in F(u, G/H)} \chi_V(g^{-1}ug) = \frac{1}{|H|} \sum_{g \in C(u, H)} \chi_V(g^{-1}ug) \]

where $C(u, H) = \{g \in G \mid g^{-1}ug \in H\}$ and $F(u, G/H) = G/H^u = \{gH \mid ugH = gH\}$. An empty sum is zero.

Proof. Since $u \in G$ sends $gH \times_H V$ to $ugH \times_H V$, we see that the direct summand $gH \times_H V$ contributes to the trace if and only if $ugH = ghH$; and in that case $l_u$ is transformed via the canonical isomorphism $i_g$ into $l_h$, $h = g^{-1}ug$. \qed
We define a linear map for class functions by the same formula

\[ \text{ind}_H^G \colon Cl(H, K) \to Cl(G, K), \quad (\text{ind}_H^G \alpha)(u) = \frac{1}{|H|} \sum_{g \in C(u, H)} \alpha(g^{-1}ug). \]

We leave it to the reader to verify the next proposition. We use the standard bilinear form \((2.6)\) on class functions. The restriction of \(\beta \in Cl(G)\) to \(H\) is given by composition with \(H \subset G\).

\[ \text{(4.1.7) Proposition. Class functions have the following properties:} \]

\[ \langle \text{ind}_H^G \alpha, \beta \rangle_G = \langle \alpha, \text{res}_H^G \beta \rangle_H, \quad \text{ind}_H^G (\alpha \cdot \text{res}_H^G \beta) = (\text{ind}_H^G \alpha) \cdot \beta, \quad \text{ind}_B^C \text{ind}_A^B = \text{ind}_A^C. \]

\[ \square \]

Problems

1. The character of \(\text{ind}_H^G V\) assumes the value 0 at \(g\) if \(g\) is not conjugate to an element of \(H\).
2. Verify directly that the assignment in \((1.1.1)\) is a homomorphism and that different choices of \(g\) lead to conjugate matrix representations.
3. Apply \((1.1.1)\) to the dihedral and quaternion groups \(D_{2n}\) and \(Q_{4n}\) and compare the result with our earlier constructions.
4. Verify \((1.1.7)\). Verify that \(\text{ind}_H^G \alpha\) is a class function.
5. Here is another dual construction of the induced representation, that is also called \textit{coinduction}. The vector space \(\text{Map}_H(G, V)\) of \(H\)-equivariant maps \(G \to V\) carries a \(G\)-action \((u \cdot \varphi)(g) = \varphi(ug)\). The decomposition of \(G\) into \(H\)-orbits, \(G = \bigsqcup Hg\), shows \(\text{Map}_H(G, V) \cong \bigoplus_{Hg} \text{Map}_H(Hg, V)\). The assignment

\[ \alpha : \text{Map}_H(G, V) \to \bigoplus gH \times H V, \quad \varphi \mapsto \sum_{Hg} (g, \varphi(g^{-1})) \]

is an isomorphism of \(G\)-representations; it sends \(\text{Map}_H(Hg, V)\) to \(g^{-1}H \times H V\).
6. The induced representation has, of course, a description in terms of modules. The regular representation \(KG\) is a left \(KG\)-module and a right \(KH\)-module. Let \(V\) be a left \(KH\)-module. Then the tensor product \(KG \otimes_{KH} V\) is a left \(KG\)-module. Relate this definition to our first definition of the induced representation; in particular explain from this view point the direct sum decomposition \((4.1)\) of the induced representation.
7. Let \(A\) and \(B\) be groups. An \((A, B)\)-set \(S\) is a set \(S\) with a left \(A\)-action and a right \(B\)-action which commute \((as)b = a(sb)), \((a, b) \in A \times B, s \in S\). Given a finite \((A, B)\)-set \(S\) we associate to an \(A\)-representation \(V\) the vector space \(\text{Map}_A(S, V)\) of \(A\)-equivariant maps \(\varphi : S \to V\). This vector space carries a \(B\)-action \((b \cdot \varphi)(s) = \varphi(sb)\). A morphism \(\alpha : V \to W\) of \(A\)-representations yields a morphism \(\text{Map}_A(S, V) \to \text{Map}_A(S, W), \varphi \mapsto \alpha \circ \varphi\). Let \(A\)-Rep denote the category of finite-dimensional left \(A\)-representations (over \(K\)). The construction above yields a functor

\[ \rho(S) : A\text{-Rep} \to B\text{-Rep} \]
for each finite \((A, B)\)-set \(S\). Let \(\gamma: S_1 \to S_2\) be a morphism of \((A, B)\)-sets. Composition with \(\gamma\) yields a morphism

\[
\rho(\gamma): \text{Map}_A(S_2, V) \to \text{Map}_A(S_1, V),
\]

and the family of these morphisms is a natural transformation \(\rho(\gamma): \rho(S_2) \to \rho(S_1)\).

Altogether we obtain a contravariant functor

\[
\rho: \text{A-Set-B} \to [\text{A-Rep}, \text{B-Rep}]
\]

of the category \(\text{A-Set-B}\) of finite \((A, B)\)-sets into the functor category.

The composition of functors \(\rho(S)\) is again a functor of the same type.

**Proposition (4.1.8)** Let \(S\) be an \((A, B)\)-set and \(T\) be a \((B, C)\)-set. Then there exists a canonical isomorphism of functors \(\rho(T) \circ \rho(S) \cong \rho(S \times_B T)\).

One has to provide a natural isomorphism

\[
\text{Map}_A(S \times_B T, V) \cong \text{Map}_B(T, \text{Map}_A(S, V))
\]

of \(C\)-representations. It will be induced by the adjunction isomorphism

\[
\text{Map}(S \times T, V) \to \text{Map}(T, \text{Map}(S, V)), \quad \varphi \mapsto \hat{\varphi}, \quad \hat{\varphi}(t)(s) = \varphi(s, t).
\]

### 4.2 Restriction to Normal Subgroups

Let \(H\) be a subgroup of \(G\). The \(g\)-conjugate \(gV\) of the \(H\)-representation \(V\) is a \(gHg^{-1}\)-representation with the same underlying vector space and with action

\[
gHg^{-1} \times V \to V, \quad (x, v) \mapsto x \cdot_{g} v = (g^{-1}xg) \cdot v.
\]

The representation \(gV\) is irreducible if and only if \(V\) is irreducible. For \(a, b \in G\) the relation \(a(bV) = abV\) holds, and \(g\)-conjugation is compatible with direct sums and tensor products. For \(h \in H\) the map \(t_h: V \to V\) is an isomorphism \(g^bV = gV\) of \(gHg^{-1}\)-representations. The bijections

\[
gH \times_H uV \to guH \times_H V, \quad (g, v) \mapsto (gu, v)
\]

combine to an isomorphism of \(G\)-representations

\[
\text{ind}^G_H uV \cong \text{ind}^G_H V.
\]

Now suppose that \(H\) is a normal subgroup of \(G\), in symbols \(H \triangleleft G\). Then \(gHg^{-1} = H\), and \(gV\) is again an \(H\)-representation which only depends on the coset \(gH\), up to isomorphism. The group \(G\) acts on the set \(\text{Irr}(H, K)\)
of isomorphism classes of $KH$-representations by $(g, V) \mapsto gV$. This action factors over an action of $G/H$ since $gV$ only depends on the coset $gH$. Let

$$G(V) = \{ g \in G \mid gV \cong V \}$$

be the isotropy group at $V$ of this $G$-action; it contains $H$.

Let $W$ be a $G$-representation and $V \subset \text{res}^G_H W$ be an $H$-sub-representation. The left translation $l_g : gV \to gV$ satisfies

$$l_g(h \cdot g v) = g(g^{-1} h g v) = h g v = h \cdot l_g v,$$

and this relation shows that $gV$ is an $H$-sub-representation which is isomorphic to $gV$ by $l_g$.

Assume, moreover, that $W$ and $V$ are irreducible. The sum of the $gV$ is a $G$-sub-representation of $W$, hence equal to $W$. Since $gV \cong V$ is irreducible, $\text{res}^G_H W$ is the sum of irreducible $H$-representations and therefore semi-simple. Thus we have shown:

**(4.2.1) Proposition.** The restriction of a semi-simple $G$-representation to a normal subgroup $H$ is a semi-simple $H$-representation.

Let again $W$ and $V \subset \text{res}^G_H W$ be irreducible. From our analysis of semi-simple representations we know that $\text{res}^G_H W$ is the direct sum of its isotypical parts, and each irreducible sub-representation of $\text{res}^G_H W$ is isomorphic to some $gV$. Let $W(V)$ be the $V$-isotypical part of $\text{res}^G_H W$. The $gV, g \in G(V)$ are contained in $W(V)$, and $W(V)$ is the sum of these $gV$. Therefore $W(V)$ is a $G(V)$-sub-representation of $W$. The inclusion $W(V) \subset \text{res}^G_H W$ gives us, by the universal property 4.1.2 of induced representations, a $G$-morphism

$$\iota : \text{ind}^G_{G(V)} W(V) \to W.$$

In our model of the induced representation, $\iota$ maps $gG(V) \times_{G(V)} W(V)$ to $gW(V)$. The subspace $gW(V)$ is another isotypical summand of $\text{res}^G_H W$. From 4.1.4 we obtain:

**(4.2.2) Proposition.** $\iota$ is an isomorphism of $G$-representations.

Suppose $W(V)$ is isomorphic to $r$ copies of $V$. Then

$$\text{res}^G_H W \cong r \bigoplus_{gG(V) \in G/G(V)} gV.$$

The $gV, gG(V) \in G/G(V)$ are pairwise non-isomorphic. The integer $r$ is called the **ramification index** of $W$ with respect to the normal subgroup $H$.

Let $V \in \text{Irr}(H; K)$. The summand $gH \times_H V$ of $\text{ind}^G_H V$ is isomorphic to $gV$; the assignment $gV \to gH \times_H V, v \mapsto (g, v)$ is an isomorphism of $H$-representations. This shows:

$$\text{res}^G_H \text{ind}^G_H V \cong |G(V)/H| \bigoplus gV.$$

**(4.4)**
The summation is over $gG(V) \in G/G(V)$. We apply Frobenius reciprocity and Schur’s lemma to (4.4) and obtain

\[
\langle \text{ind}_{G/H}^G V, \text{ind}_{G/H}^G V \rangle_G = |G(V)/H| \sum \langle V, gV \rangle_H = |G(V)/H| \langle V, V \rangle_H.
\] (4.5)

From this relation we deduce:

**4.2.3 Proposition.** Let $K$ be algebraically closed of characteristic not dividing $|G|$. Then $\text{ind}_{G/H}^G V$ is irreducible if and only if $G(V) = H$.

**Proof.** $\text{ind}_{G/H}^G V = W$ is irreducible if and only if $\langle W, W \rangle_G = 1$, and this is, by 4.5, the case if and only if $|G(V)/H| = 1$ and $\langle V, V \rangle_H = 1$. \qed

**4.2.4 Theorem.** Let $K$ be algebraically closed of characteristic zero. Let $H \triangleleft G$ and $V \in \text{Irr}(H; K)$. Suppose $\text{ind}_{G/H}^G(V) V = \bigoplus_{j=1}^k m_j V_j$ with pairwise non-isomorphic $G(V)$-representations $V_j$. Then:

1. $\text{ind}_{G(V)/H} G(V) V_j$ is irreducible.
2. Let $W \in \text{Irr}(G; K)$ and $\langle V, \text{res}_{G/H}^G W \rangle \neq 0$. Then $W \cong W_j$ for some $j$.
3. The $W_j$ are pairwise non-isomorphic.
4. $m_j$ is the ramification index of $W_j$ with respect to $H$.
5. Let $I^G(V) = \{W_1, \ldots, W_r\} \subset \text{Irr}(G; K)$ and $\text{Irr}(G; K)$ is the disjoint union of the sets $I^G(V)$ where $V$ runs through a representative system of conjugation orbits $\text{Irr}(H; K)/G$.

**Proof.** Restriction to $H$ and (4.4) yields

\[
\bigoplus_{j=1}^k m_j \text{res}_{G/H}^G V_j = \text{res}_{G/H}^G \text{ind}_{G/H}^G V = |G(V)/H| V.
\] (4.6)

Therefore $\text{res}_{G/H}^G V_j = n_j V$ for some $n_j \in \mathbb{N}$. Frobenius reciprocity yields

\[
n_j = \langle n_j V, V \rangle_H = \langle V, \text{res}_{G/H}^G V_j \rangle_H = \langle \text{ind}_{G/H}^G V, V_j \rangle_G = m_j \langle V_j, V \rangle_{G(V)} = m_j.
\]

Therefore $\text{res}_{G/H}^G V_j = m_j V$, and together with (4.6) we obtain

\[
|G(V)/H| = \sum_{j=1}^k m_j^2.
\] (4.7)

Let now $W \in \text{Irr}(G, K)$ be such that

\[
\langle \text{ind}_{G(V)/H}^G V, W \rangle_G \neq 0.
\] (4.8)

We want to show that $W \cong W_i = \text{ind}_{G(V)/H}^G V_i$; this shows in particular that $W_i$ is irreducible, since there exist $W$ such that (4.8) holds. By Frobenius
reciprocity \( \langle \text{ind}_{G(V)} G(V), W \rangle_G = \langle V, \text{res}_{G(V)} G(V) W \rangle_{G(V)} \). Therefore \( V_i \) occurs in \( \text{res}_{G(V)} G(V) W \) and hence \( \langle U, V_i \rangle \leq \langle U, \text{res}_{G(V)} G(V) W \rangle \) for each \( G(V) \)-representation \( U \). In particular

\[
\langle V, \text{res}_{G(V)} G(V) W \rangle = \langle \text{ind}_{H} G(V) V, \text{res}_{G(V)} G(V) W \rangle \\
\geq \langle \text{ind}_{H} G(V) V, V_i \rangle = \langle V, \text{res}_{H} G(V) V_i \rangle = m_i.
\]

This implies, for each \( g \in G \),

\[
\langle \text{res}_{H} G(V) W, 9V \rangle = \langle \text{res}_{H} G(V) (g^{-1} W), V \rangle = \langle \text{res}_{H} G(V) W, V \rangle \geq m_i.
\]

Therefore each \( 9V \) occurs in \( \text{res}_{H} G(V) W \) at least with multiplicity \( m_i \). From (4.8) we obtain

\[
\dim W \leq \dim \text{ind}_{H} G(V) V_i. \tag{4.9}
\]

Since \( 9V \) occurs in \( \text{res}_{H} G(V) W \) at least with multiplicity \( m_i \) and since there exist \( |G/G(V)| \) different \( 9V \), we see

\[
\dim W \geq m_i |G/G(V)| \dim V = |G/G(V)| \dim V_i = \dim \text{ind}_{H} G(V) V_i. \tag{4.10}
\]

From (4.9) and (4.10) we obtain equality of dimensions and therefore \( W \cong \text{ind}_{H} G(V) V_i \), since \( W \) occurs in \( \text{ind}_{H} G(V) V_i \). Frobenius reciprocity again yields

\[
\langle V, \text{res}_{H} G(V) W \rangle = \langle \text{ind}_{H} G(V) V, W \rangle = \sum_j m_j \langle \text{ind}_{G(V)} G(V) V_j, W \rangle.
\]

Therefore \( W \in \text{Irr}(G, K) \) is of the form \( \text{ind}_{G(V)} G(V) V_j \) for some \( j \in \{1, \ldots, k\} \) if and only if \( \langle V, \text{res}_{H} G(V) W \rangle \neq 0 \). There are exactly \( k \) \( G \)-representations of this type if we show \( W_i \not\cong W_j \) for \( i \neq j \). Suppose that \( W_1 \cong W_2 \). Then

\[
\text{ind}_{H} G(V) V = (m_1 + m_2) \text{ind}_{G(V)} G(V) V_2 + m_3 \text{ind}_{G(V)} G(V) V_3 + \cdots \tag{4.11}
\]

and, together with (4.7) and 4.2.3, we arrive at the contradiction

\[
\sum_j m_j^2 = |G(V)/H| = \langle \text{ind}_{H} G(V) V, \text{ind}_{H} G(V) V \rangle \geq (m_1 + m_2)^2 + m_3^2 + \cdots + m_k^2;
\]

the inequality \( \geq \) is a consequence of (4.11) and \( \langle W_i, W_i \rangle = 1 \).

The ramification index of \( W_j \) with respect to \( H \) is \( \langle V, \text{res}_{H} G(V) W_j \rangle \). By Frobenius reciprocity this is equal to \( \langle \text{ind}_{H} G(V) V, W_j \rangle = \langle \sum_j m_i W_i, W_j \rangle = m_j \).

Suppose \( W \in I^G(V_1) \cap I^G(V_2) \). Then

\[
0 \neq \langle \text{ind}_{H} G(V_1) V_1, \text{ind}_{H} G(V_2) V_2 \rangle = \langle V_1, \text{res}_{H} G(V_2) \rangle.
\]

Since \( \text{res}_{H} G(V) V_2 \) contains only conjugates of \( V_2 \), we see that \( V_1 \) and \( V_2 \) are conjugate. Part (2) and \( I^G(V) = I^G(9V) \) now shows that \( \text{Irr}(G; K) \) is the disjoint union of the \( I^G(V) \) as stated. \( \square \)
### 4.2.5 Remark. Theorem 4.2.4 gives a kind of recipe for the construction of irreducible $G$-representations starting from the irreducible representations of a normal subgroup $H$.

The situation is easy to survey if $V$ happens to be a restriction of a $G(V)$-representation $\tilde{V}$. In that case $\text{ind}_{H}^{G(V)}V = \text{ind}_{H}^{G(V)}1_{H} \otimes \tilde{V}$, see ???. Since $H \leq G(V)$, the representation $\text{ind}_{H}^{G(V)}1_{H}$ is obtained from the regular representation $K(G(V)/H)$ by composition with the quotient homomorphism $G(V) \to G(V)/H$. The decomposition of the regular representation now determines the $V_j$ in 4.2.4. \hfill \Box

The next proposition gives conditions under which the lifting property holds for all irreducible representations of $H$. For the proof go back to 1.6.1.

#### (4.2.6) Proposition. Let $K$ be algebraically closed. Suppose $H$ is an abelian normal subgroup and $G$ the semi-direct product of $H$ and $P$, i.e. $G = HP$ and $H \cap P = 1$. Then each irreducible $H$-representation $V$ has an extension to $G(V)$.

#### (4.2.7) Remark. We now combine 4.2.4 - 4.2.6. The hypotheses are as in 4.2.6. The irreducible representations of $G$ are obtained as follows. Start with $\eta \in \text{Irr}(H)$. Let $P_\eta \leq P$ be the isotropy group of $\eta$ under the conjugation action of $P$ on $\text{Irr}(H)$. Extend $\eta$ to $\tilde{\eta}$ by 4.2.6. Let $U \in \text{Irr}(P_\eta)$ and lift to a representation $\tilde{U}$ of $HP_\eta$. Then form $W = \text{ind}_{HP_\eta}(\tilde{\eta} \otimes \tilde{U})$. The isomorphism class of $W$ uniquely determines the $P_\eta$-orbit of $\eta$ and the isomorphism class of the $P_\eta$-representation $U$. \hfill \Box

### 4.3 Monomial Groups

The induced representation of a one-dimensional representation is called a monomial representation. A group is called monomial if each $V \in \text{Irr}(G; \mathbb{C})$ is monomial.

Let $\rho: H \to K^*$ be a one-dimensional representation. A basis of $\text{ind}_{H}^{G}\rho$ consists of the $(g_j, 1) = x_j$ where $1 = g_1, \ldots, g_r$ is a representative system of $G/H$. Suppose $gg_jg = g_{\sigma(j)}h_j$ with $\sigma \in S_r$ and $h_j \in H$. The computation $gx_j = (gg_j, 1) = (g_{\sigma(j)}h_j, 1) = (g_{\sigma(j)}, \rho(h_j)) = \rho(h_j)x_{\sigma(j)}$

shows: The matrix representation of $\text{ind}_{H}^{G}\rho$ with respect to the basis above consists of matrices which have in each row and column exactly one non-zero entry. Matrices of this type are called monomial.

A group $G$ is said to be supersolvable if there exists a string of normal subgroups $1 = G_0 < G_1 < \ldots < G_r = G$ such that $G_j/G_{j-1}$ is a group of
prime order. Subgroups and factor groups of supersolvable groups are supersolvable. Groups of prime power order are supersolvable. If $H$ is cyclic and $G/H$ supersolvable, then $G$ is supersolvable.

(4.3.1) Theorem. Supersolvable groups are monomial.

The proof needs some preparation and will be finished after 4.3.4.

(4.3.2) Proposition. Suppose $G$ has an abelian normal subgroup $A$ which is not central. Then a faithful irreducible $\mathbb{C}G$-representation is induced from a proper subgroup.

Proof. Let $W \in \text{Irr}(G; \mathbb{C})$ be faithful and suppose $V \in \text{Irr}(H; \mathbb{C})$ is an $H$-sub-representation of $W$. It suffices to show $G(V) \neq G$; see 4.2.2. Suppose $G(V) = G$. Then $\text{res}^G_A W$ is a multiple of the one-dimensional representation $V$. Therefore each $a \in A$ acts on $V$ as multiplication by a scalar, hence $l_a$ commutes with $l_g$ for each $g \in G$. Since $W$ is faithful this fact implies that $a$ is contained in the center of $G$.

(4.3.3) Lemma. Let $G$ be a non-abelian supersolvable group. Then $G$ contains a non-central normal abelian subgroup.

Proof. Let $Z < G$ be the center of $G$. Since $G$ is supersolvable, so is $G/Z$. Let $1 \neq H/Z \unlhd G/Z$ be a cyclic normal subgroup. Then $H$ is an abelian non-central normal subgroup.

We need a formal property of induced representations. Let $\alpha: A \to B$ be a homomorphism. We associate to a $KB$-representation $V$ a $KA$-representation $\alpha^* V$ with the same underlying vector space $V$ and with action

$$A \times V \to V, \quad (a, v) \mapsto \alpha(a) \cdot v.$$ 

In the case that $\alpha: A \subset B$ we have $\alpha^* V = \text{res}^B_A V$. If $\alpha$ is surjective, we say that $\alpha^* V$ is obtained from $V$ by lifting the group action along $\alpha$. We show that induction is compatible with lifting. Consider

$$\tilde{\alpha}^{-1}(A) = P \overset{\alpha}{\longrightarrow} Q \overset{\tilde{\beta}}{\longrightarrow} A \overset{\subset}{\longrightarrow} B,$$

$\tilde{\alpha}$ surjective and $\alpha = \tilde{\alpha}|P$. Then

(4.3.4) Lemma. $\tilde{\alpha}^*(\text{ind}_P^B V) \cong \text{ind}_P^B(\alpha^* V)$ for each $A$-representation $V$.

Proof. $\alpha$ induces a bijection $Q/P \to B/A$ by passing to quotients. The isomorphism is induced by $qP \times_P \alpha^* V \to \alpha(q)A \times_A V, \quad (q, v) \mapsto (\alpha(q), v)$. 

□
Proof. (Of 4.3.1.) Let $W$ be a faithful irreducible \( CG \)-representation. If $G$ is abelian, then $\dim W = 1$, and nothing is to prove. Otherwise $W$ is, by 4.3.2 and 4.3.3, induced from a proper subgroup $W = \text{ind}_H^G V$. By induction we can assume that $V$ is monomial, and by transitivity of induction, $W$ is monomial.

If $W$ is not faithful, let $L$ be its kernel, and consider $W$ as $G/L$-representation $\overline{W}$. By induction again, $\overline{W}$ is monomial. Now use 4.3.4.

Problems

1. Consider the semi-direct product $G$ of the quaternion group $Q_8 = \langle a, b \mid a^2 = b^2, bab^{-1} = a^{-1} \rangle$ by $C_3 = \langle h \rangle$ with respect to the automorphism $hah^{-1} = b, hbh^{-1} = ab$. The faithful representation $G \to SU(2)$

\[
\begin{align*}
a &\mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
b &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
h &\mapsto \frac{i-1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
\end{align*}
\]

is not monomial: The group $G$ has no subgroup of index 2. The group $G$ has the quotient $A_4$ and is solvable. ($G$ is called the binary tetrahedral group.)

4.4 The Character Ring and the Representation Ring

Let $K$ be a field of characteristic zero. Recall that the characters of the irreducible $KG$-representations are linearly independent in the ring of class functions $Cl(G; K)$. The additive subgroup $CH(G; K)$ of $Cl(G; K)$ generated by the characters of irreducible representations is therefore a free abelian group of rank $|\text{Irr}(G; K)|$. The relation $\chi_{V \oplus W} = \chi_V + \chi_W$ shows that each character is contained in this group. And the relation $\chi_{V \otimes W} = \chi_V \chi_W$ is used to show that $CH(G; K)$ is a subring of $Cl(G; K)$. This ring is called the character ring of $KG$-representations.

The character ring can be constructed formally. It is then called the representation ring or Green ring. In this context $K$ can be an arbitrary field. Let $R(G; K)^+$ denote the set of isomorphism classes of $KG$-representations. Direct sum and tensor product induces on $R(G; K)^+$ two composition laws (addition and multiplication), and with these structures $R(G; K)^+$ is almost a commutative ring, except that inverses for the additive structure are missing. In situations like this, there exists a universal ring $R(G; K)$ together with a homomorphism

\[
i: R(G; K)^+ \to R(G; K)
\]

of semi-rings which is determined, up to a unique isomorphism, by a universal property: Let $\varphi: R(G; K)^+ \to A$ be a homomorphism into an abelian group.
Then there exists a unique homomorphism of abelian groups $\Phi: R(G; K) \to A$ such that $\Phi \circ \iota = \varphi$. If, moreover, $A$ is a commutative ring and $\varphi$ a homomorphism of semi-rings, then the universal homomorphism $\Phi$ is a homomorphism of rings. The universal property is used to show that additive constructions with representations extend to the representation ring. Elements in $R(G; K)$ are formal differences $[V] - [W]$ of representations $V, W$, called **virtual representations**, and $[V] - [W] = [V'] - [W']$ if and only if $V \oplus W' \oplus Z \cong V' \oplus W \oplus Z$ for some representation $Z$.

**Example.** Let $G = C_m = \langle a \mid a^m = 1 \rangle$ and $\rho: a \mapsto \exp(2\pi i/m)$ the standard representation. Then $R(G; \mathbb{C})$ is the free abelian group with basis $1, \rho, \rho^2, \ldots, \rho^{m-1}$. The multiplicative properties of the $\rho^k$, namely $\rho^k \otimes \rho' \cong \rho^{k+l}$, show that the ring $R(G; \mathbb{C})$ is isomorphic to $\mathbb{Z}[\rho]/(\rho^m - 1)$. More formally: For a finite abelian group $G$ with character group $G^*$ the representation ring $R(G; \mathbb{C})$ is isomorphic to the group ring $\mathbb{Z}[G^*]$. 

**Example.** We determine $R(\mathbb{C}; \mathbb{Q})$ for $\mathbb{C} = \langle x \mid x^m = 1 \rangle$. Decompose $x^m - 1 \in \mathbb{Q}[x]$ into irreducible factors $x^m - 1 = \prod_{d|m} \Phi_d(x)$. The cyclotomic polynomial $\Phi_d(x)$ has the primitive $d$-th roots of unity as its roots. The quotient $V_d = \mathbb{Q}[x]/(\Phi_d(x))$, viewed as a module over the ring $\mathbb{Q}C_m = \mathbb{Q}[x]/(x^m - 1)$, is an irreducible $\mathbb{Q}C_m$-representation. The $V_d, d|m$ form a $\mathbb{Z}$-basis of $R(\mathbb{C}; \mathbb{Q})$. There is another $\mathbb{Z}$-basis which consists of the permutation representations $\mathbb{Q}(C_m/C_n) = P_{m/n}$. The representation $P_{m/n}$ contains the irreducible representations which have $C_n$ in its kernel. The kernel of $V_d$ is $C_{m/d}$. Hence $P_k = \sum_{d|k} V_d$. By Möbius-inversion one obtains $V_k = \sum_{d|k} \mu(k/d)P_d$.

**Example.** Suppose the characteristic of $K$ does not divide the order of $G$. We describe $R(\mathbb{C}; K)$. Let $L = K(\varepsilon)$ be the field extension, $\varepsilon$ a primitive $m$-th root of unity. Then, as in [4.4.3], $R(\mathbb{C}; L) \cong \mathbb{Z}[\rho]/(\rho^m - 1)$ where $\rho: C_m \to L^*$ is given by $\rho(a) = \varepsilon$. Field extension yields an injective homomorphism $\iota: R(C_m; K) \to R(C_m; L)$. Let $\Gamma = \text{Gal}(L|K)$ be the Galois group of $L$ over $K$. An element $\gamma \in \Gamma$ is determined by its value $\gamma(\varepsilon) = \varepsilon^t$; and
4.5 Cyclic Induction

Let $F$ be a set of subgroups of $G$. A general question of induction theory is: For which sets $F$ is the induction map

$$i_F = \langle \text{ind}_H^G | H \in F \rangle: \bigoplus_{H \in F} R(H; K) \rightarrow R(G; K)$$

surjective? We know that the image of $i_F$ is an ideal. Therefore $i_F$ is surjective if and only if the unit element $1_G$ of $R(G; K)$ is contained in the image of $i_F$. It is also interesting to look for integral multiples of $1_G$ in the image of $i_F$.

Let $K$ be a field of characteristic zero. In this section we prove Artin’s induction theorem which says that $|G|1_G$ is in the image of $i_F$ for the set $C$ of cyclic subgroups. The proof is based on a character calculation.

We rewrite the basic orthogonality relation

$$|G| \dim V^G = \sum_{g \in G} \chi_V(g). \quad (4.12)$$
Let $C^\#$ denote the set of generators of the cyclic group $C$. Since each element generates a unique cyclic subgroup we can write the right hand side of (4.12) as a sum over the cyclic subgroups $C$ of $G$

\[
|G| \dim V^G = \sum_{C}(\sum_{g \in C^\#} \chi_V(g)). \tag{4.13}
\]

We apply this to the cyclic subgroup $C$ itself.

Let $\mu : \mathbb{N} \to \mathbb{Z}$ be the Möbius-function, defined inductively by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for $n > 1$. Let $f$ and $g$ be functions from $\mathbb{N}$ into some (additive) abelian group such that $f(n) = \sum_{d|n} g(d)$; then Möbius inversion tells us that $g(n) = \sum_{d|n} \mu(n/d) f(d)$.

Note that for each divisor $d$ of $|C| = n$ there exists a unique subgroup $D \leq C$ with $|D| = d$. We obtain by Möbius inversion from (4.13)

\[
\sum_{c \in C^\#} \chi_V(c) = \sum_{D \leq C} \mu(|C/D|)(\sum_{d \in D} \chi_V(d)). \tag{4.14}
\]

The inner sum in (4.14) equals $|D| \dim V^D$. Therefore we obtain altogether:

\[
|G| \dim V^G = \sum_{C}(\sum_{D \leq C} \mu(|C/D|)|D| \dim V^D). \tag{4.15}
\]

This equality has the form $|G| \dim V^G = \sum_{C} a_C |K(G/C)|$ with suitable integers $a_C$. We use $(K(G/H), V)_G = \dim V^H$ in (4.15) and see that $\langle |G|K(G/G) - \sum_{C} a_C K(G/C), V \rangle_G = 0$ for each representation $V$. This implies that the left argument of the bracket is zero.

(4.5.1) Proposition. $|G|[K(G/G)] = \sum_{C} a_C |K(G/C)|$ in $R(G; K)$. Note that $1_G = [K(G/G)]$ is the unit element in $R(G; K)$. \hfill $\square$

We know that $\text{ind}^G_C : R(G; K) \to R(G; K)$ is multiplication by $[K(G/C)]$, see (4.1.5). Hence we obtain from (4.5.1)

(4.5.2) Theorem. For $x \in R(G; K)$ the identity \[
\sum_{C} a_C \text{ind}^G_C x = |G|x
\]
holds. This implies Artin’s induction theorem: $|G|R(G; K)$ is contained in the image of $i_C$. \hfill $\square$

(4.5.3) Theorem. Let $V$ and $W$ be $\mathbb{Q}G$-representations. Suppose that for each cyclic subgroup $C \subset G$ we have $\dim V^C = \dim W^C$. Then $V$ and $W$ are isomorphic.

Proof. It suffices to show $\text{res}^G_C V \cong \text{res}^G_C W$ for each cyclic subgroup $C$ of $G$, since we know that representations are determined by their restriction to cyclic subgroups. From our analysis of irreducible $\mathbb{Q}C$-representations we conclude that they are determined up to isomorphism by fixed point dimensions of subgroups. \hfill $\square$

\[^{1}d|n\] means, $d$ divides $n$.\n
(4.5.4) Proposition. The rank of $R(G; \mathbb{Q})$ equals the number of conjugacy classes of cyclic subgroups of $G$.

Proof. Since conjugate subgroups have fixed points of equal dimension we conclude from (4.5.3) that the rank is at most the number of cyclic conjugacy classes.

We show that the permutations representations $U_C = \mathbb{Q}(G/C)$, $(C)$ cyclic conjugacy class, are linearly independent. Suppose $\sum_{(C)} a_C U_C = 0$. Let $C$ be maximal such that $a_C \neq 0$, and let $g \in C$ be a generator. The character of $\sum_{(C)} a_C U_C$ at $g$ is $a_C |G/C| \neq 0$; a contradiction.

(4.5.5) Proposition. Let $\alpha \in \text{Cl}(G; K)$. Suppose for each cyclic subgroup $C$ the restriction $\text{res}_G^C \alpha \in R(C; K)$. Then $|G| \alpha \in R(G; K)$.

Proof. This is a direct consequence of (4.5.2).

Problems

1. Let $G = A_5$. The virtual permutation representation associated to $[G/C_5] + [G/C_3] + [G/C_2] - [G/C_1]$ realizes $2 \cdot 1_G \in R(G; \mathbb{Q})$.

4.6 Induction Theorems

In order to state further induction theorems we have to specify suitable sets of subgroups. Let $p$ be a prime number. A $p$-group is a group of $p$-power order. Let $|G| = p^t q^s$ with $(p, q) = 1$. Then there exists a subgroup $G(p) \leq G$ of order $p^t$, and all such groups are conjugate; they are called Sylow $p$-groups of $G$. A $p$-hyperelementary group $H$ is the semi-direct product of a cyclic group and a $p$-group $P$ of coprime order; $S$ is a normal subgroup of $H$ and $H/S \cong P$. Let $\mathcal{H}(p, G)$ denote the set of $p$-hyperelementary subgroups of $G$. The set $\mathcal{H}(G) = \bigcup_{p} \mathcal{H}(p, G)$ is the set of hyperelementary subgroups of $G$. Hyperelementary groups are monomial. A $p$-elementary group $H$ is the direct product $S \times P$ of a cyclic group $S$ and a $p$-group $P$ of coprime order. We denote by $\mathcal{E}(p, G)$ the set of $p$-elementary subgroups of $G$ and by $\mathcal{E}(G) = \bigcup_{p=1}^n \mathcal{E}(p, G)$ the set of elementary subgroups of $G$.

As in the case of Artin’s induction theorem, the hyperelementary induction theorem is a consequence of a result about permutation representations.

(4.6.1) Theorem. Let $K$ be a field of characteristic zero. There exists in $R(G; K)$ a relation of the type $|G/G(p)| 1_G = \sum_{E \in \mathcal{H}(p, G)} h_E[K(G/E)]$ with suitable integers $h_E$. 

We defer the proof to a later section where we deal systematically with such results; see ??.

(4.6.2) Theorem (Hyper-elementary induction). Let $K$ be a field of characteristic zero. Then $|G/G(p)| R(G; K)$ is contained in the image of $i_H(p, G)$. The induction map $i_{H(G)}$ is surjective.

Proof. The first assertion is a consequence of 4.6.1. The integers $|G/G(p)|, p$ a divisor of $|G|$, have no common divisor. Hence there exist integers $n_p$ such that $\sum_p n_p |G/G(p)| = 1$. We conclude that $1_G$ is in the image of $i_{H(G)}$. \[\square\]

Hyper-elementary groups are supersolvable. Therefore the next theorem is a consequence of 4.6.2.

(4.6.3) Theorem (Monomial induction). $R(G; C)$ is generated by monomial representations, i.e., each element $x \in R(G; C)$ is of the form $x = [V] - [W]$ with representations $V$ and $W$ which are direct sums of monomial representations. \[\square\]

(4.6.4) Proposition. Let $H = SP$ be a $p$-hyper-elementary group. Then $i_E(p, H)$ is surjective $(K = C)$.

Proof. By induction on $|H|$ we can assume that irreducible representations of dimension greater than one are in the image of the induction map. Let $\alpha \in X(H)$ be a one-dimensional representation. Consider the elementary subgroup $E = S^p \times P$. We claim: If $\gamma \in X(H)$ occurs in $\text{ind}^{G}_{E} \text{res}^{G}_{E} \alpha$, then $\alpha = \gamma$ and $\alpha$ occurs with multiplicity one. By Frobenius reciprocity this is a consequence of 1.6.3. Thus modulo representations of dimensions greater than one, each element of $X(H)$ is in the image of the induction map. \[\square\]

We combine 4.6.2 and 4.6.4 and obtain:

(4.6.5) Theorem (Brauer’s induction theorem). Let $K = C$. Then $|G/G(p)| R(G; C)$ is contained in the image of $i_{E(p, G)}$, and $i_{E(G)}$ is surjective. \[\square\]

We now derive an interesting consequence of the monomial induction theorem. The exponent $e(G)$ of a group $G$ is the least common multiple of the order of its elements; it divides $|G|$.

(4.6.6) Theorem (Splitting field). Let $\epsilon$ be a primitive $e(G)$-th root of unity. Then $\mathbb{Q}(\epsilon)$ is a splitting field for $G$, i.e., each irreducible $CG$-representation has a realization with matrices having entries in $\mathbb{Q}(\epsilon)$.

Proof. One-dimensional representations of subgroups of $G$ are certainly realizable over $\mathbb{Q}(\epsilon)$ and therefore also monomial representations, being induced from one-dimensional ones. From 4.6.3 we infer: Let $M$ be a $CG$-representation.
Then there exist representations $U$ and $V$, which are realizable over $\mathbb{Q}(\varepsilon)$ and such that $U \cong V \oplus M$. It is now a general fact that this implies: $M$ is realizable over $\mathbb{Q}(\varepsilon)$. See the next proposition.

Let $L$ be an extension field of $K$ (of characteristic zero). We denote by $V_L = L \otimes_K V$ the extension of a $KG$-representation to an $LG$-representation. From character theory we see $\langle V, W \rangle_{KG} = \langle V_L, W_L \rangle_{LG}$.

**Proposition.** Let $U$ and $V$ be $KG$-representations and $M$ an $LG$-representation. Suppose $U_L \cong V_L \oplus M$. Then there exists a $KG$-representation $N$ such that $N_L \cong M$.

**Proof.** We fix $M$. Among all possible isomorphisms $U_L \cong V_L \oplus M$ choose one with $V$ of smallest dimension. We show $V = 0$. Suppose $V \neq 0$. Choose $W \in \text{Irr}(G; K)$ with $\langle W, V \rangle_{KG} > 0$. Then $\langle U, W \rangle_{KG} = \langle U_L, W_L \rangle_{LG} = \langle V_L \oplus M, W_L \rangle_{LG} = \langle V, W \rangle_{KG} + \langle M, W_L \rangle_{LG} > 0$.

Therefore $W$ occurs in $U$, hence $U \cong U' \oplus W$, $V \cong V' \oplus W$ by semi-simplicity. We conclude $U'_L \oplus W_L \cong V'_L \oplus W_L \oplus M$, cancel $W_L$, and see that $V$ was not of minimal dimension.

**Proposition.** The restriction $\rho: R(G; \mathbb{C}) \rightarrow \prod_E R(E; \mathbb{C})$ is an injection as a direct summand ($E$ elementary subgroups).

**Proof.** Suppose $1_G = \sum_E \text{ind}_E^G x_E$. Define $\lambda: R(G) \rightarrow \prod_E R(E), \quad x \mapsto (\text{res}_E^G x : x_E \mid E)$. Then $\sum_E \text{ind}_E^G (\text{res}_E^G x : x_E) = \sum_E x \cdot \text{ind}_E^G x_E = x \cdot 1_G = x$. Hence $\lambda$ is a splitting of the induction.

Dually, define $r: \bigoplus_E R(E) \rightarrow R(G), \quad (y_E) \mapsto \sum_E \text{ind}_E^G (y_E \cdot x_E)$. Then $r \rho(x) = \sum \text{ind}_E^G (\text{res}_E^G x \cdot x_E) = x \cdot \sum E \text{ind}_E^G x_E = x$. Thus $r$ is a splitting of $\rho$.

**Proposition.** Let $\alpha \in \text{Cl}(G)$ be such that $\text{res}_E^G \alpha \in R(E)$ for each elementary subgroup $E$ of $G$. Then $\alpha \in R(G)$.

**Proof.** This is a consequence of 4.6.7.

For arbitrary fields (of characteristic zero) we have results of the type 4.6.7 and 4.6.8 using hyperelementary groups. The proofs are the same.
Problems

1. Verify the elementary induction theorem explicitly for \( G = A_5 \).
2. The virtual permutation representation associated to \( [G/D_5] + [G/D_3] - [G/D_2] \) is the unit element, \( G = A_5 \).

4.7 Elementary Abelian Groups

We assume that \( p \) does not divide the characteristic of \( K \).

\( 4.7.1 \) Theorem. Let \( V \) be a faithful \( KG \)-representation and \( A \triangleleft G \) an elementary abelian group of rank \( n \geq 2 \). Then there exists \( H \leq A \) such that \( \lfloor A/H \rfloor = p \) and \( V^H \neq 0 \). The normalizer \( N_GH \) is different from \( G \) and the canonical map \( \text{ind}^G_{N_GH} V^H \to V \) is an isomorphism.

The proof needs some preparation. Let \( S(A) = \{ H \leq A \mid \lfloor A/H \rfloor = p \} \) be the set of cocyclic subgroups. Consider the following elements in the group algebra \( KA \)

\[
x_H = |A|^{-1}(p\Sigma_H - \Sigma_A), \quad y = |A|^{-1}\Sigma_A
\]

with \( \Sigma_A = \sum_{a \in A} \) and \( \Sigma_H = \sum_{h \in H} h \) for \( H \in S(A) \).

\( 4.7.2 \) Proposition. The elements \( x_H \) and \( y \) have the following properties:

1. \( x_H^2 = x_H \); \( y^2 = y \)
2. \( x_Hx_K = 0 \) for \( H \neq K \); \( x_Hy = 0 \)
3. \( 1 = y + \sum_{H \in S(A)} x_H \).

Proof. The proof of (1) and (2) is a direct consequence of the relations \( \Sigma_A^2 = |A|\Sigma_A \), \( \Sigma_H\Sigma_A = |H|\Sigma_A \), \( \Sigma_H^2 = \Sigma_H \) and \( \Sigma_H\Sigma_K = p^{-2}|A|\Sigma_A \) for \( H \neq K \). In order to prove (3), one has to count the number of \( H \in S(A) \) which contain \( 1 \neq a \) and the cardinality of \( S(A) \). The latter equals the number \( (p^n - 1)/(p - 1) \) of one-dimensional subspaces of \( A \). The former is the number of subspaces of \( A/\langle a \rangle \) of codimension one. From this information one verifies (3).

\( 4.7.3 \) Proposition. Let \( V \) be a \( KA \)-representation. Then \( yV = Y^A \) and \( x_HV \oplus V^A = V^H \).

Proof. We already know that multiplication with \( y \) is a projection operator onto the fixed point set. The second assertion follows from \( |H|^{-1}\Sigma_H = x_h + y \), \( x_Hy = 0 \), and the fact that multiplication by \( |H|^{-1}\Sigma_H \) is the projection onto \( V^H \).

\( 4.7.4 \) Proposition. \( V = yV \oplus \bigoplus_{H \in S(A)} x_HV \).
Proof. \ref{4.7.2} (3) shows that $V$ is the sum of $gV$ and the $xHV$, and \ref{4.7.2} (2) shows that the sum is direct.

\textbf{(4.7.5) Corollary.} Let $S(A,V) = \{ H \in S(A) \mid V^H \neq 0 \}$ and suppose $V^G = 0$. Then $V = \bigoplus_{H \in S(A,V)} V^H$. In particular $S(A,V) \neq \emptyset$.

Proof. (Of \ref{4.7.1}). By \ref{4.7.4} there exists $H \in S(A)$ such that $V^H \neq 0$. Since $gV^H = V^{gHg^{-1}}$, the group $G$ acts on $S(A,V)$ by conjugation. Since $\sum gV^H = V$, the action is transitive, and $NH$ is the isotropy group of $H \in S(A,V)$. The statement is now a special case of \ref{4.2.1}.

We report on groups to which \ref{4.7.1} applies.

\textbf{(4.7.6) Theorem.} Suppose each abelian normal subgroup of the $p$-group $G$ is cyclic. Then $G$ is a group in the following list.

1. $G$ is cyclic.
2. $G$ is the dihedral group $D(2^n)$ of order $2^n$, $n \geq 4$. It has the presentation $\langle A, B \mid A^{2^n-1} = 1 = B^2, BAB^{-1} = A^{-1} \rangle$.
3. $G$ is the semi-dihedral group $SD(2^n)$ of order $2^n$, $n \geq 4$. It has the presentation $\langle A, B \mid A^{2^n-1} = 1 = B^2, BAB^{-1} = A^{2^{n-2}-1} \rangle$.
4. $G$ is the quaternion group $Q(2^n)$ of order $2^n$, $n \geq 3$. It has the presentation $\langle A, B \mid B^2 = A^{2^n-2}, BAB^{-1} = A^{-1} \rangle$.

We have shown that complex representations of $p$-groups are induced from one-dimensional representations. The virtue of \ref{4.7.1} is that we do not need any hypothesis about the field $K$, except that we are in the semi-simple case. Thus \ref{4.7.1} applies, e.g., to real or rational representations. It then remains to study the groups in the list \ref{4.7.5}. The situation is particularly simple for $p \neq 2$, since then everything is reduced to the cyclic groups.