A non obvious estimate for the pressure

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Abstract
In Euler and Navier Stokes equations, the pressure is related to the velocity by the formula
\[ p = R_i R_j u_i u_j. \] We prove that if \( u \in C^\alpha \) then \( p \in C^{2\alpha} \).

1 Introduction

In Euler or Navier Stokes equation, the pressure is computed from the velocity by the formula
\[ p = R_i R_j u_i u_j. \quad (1.1) \]
where \( R_j \) denotes the Riesz transform and repeated indexes are summed. Since the Riesz transforms are operators of order zero, it is generally understood that \( p \) would have the same regularity estimates as \( u \otimes u \) or \( |u|^2 \). Therefore, if \( u \in C^\alpha \), it is natural to obtain that also \( p \in C^\alpha \). The purpose of this note is to show that if \( \alpha \in (0, 1/2) \cup (1/2, 1) \), actually \( p \in C^{2\alpha} \), which seems somewhat surprising.

The case \( \alpha = 1/2 \) is a borderline case because in that case one would expect \( p \) to be Lipschitz. It is well known that that kind spaces do not get along well with singular integrals.

Note that (1.1) arises from the following equivalent formula
\[ \Delta p = \partial_i \partial_j u_i u_j. \quad (1.2) \]
Even thought the most interesting cases for Euler or Navier Stokes equation are in dimension 2 and 3, we will present the proof in arbitrary dimension \( n \), since there is no difference in difficulty.

As a notational clarification, we denote by \( [u]_{C^\alpha} \) the \( C^\alpha \) seminorm given by
\[ [u]_C^\alpha = \sup_{x,y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{1+\alpha}}. \]

The main result of this note is the following.

**Theorem 1.1.** Assume \( u \in C^\alpha \) for \( \alpha \in (0, 1/2) \cup (1/2, 1) \) is a divergence free vector field, and \( p \) be given by the formula (1.1). Then if \( \alpha \in (0, 1/2) \), we have for all \( x, y \in \mathbb{R}^n \),

\[ |p(x) - p(y)| \leq C|x - y|^{2\alpha}[u]_{C^\alpha}^2, \]

where \( C \) is a constant depending on \( n \) and \( \alpha \). In addition, if \( \alpha \in (1/2, 1) \),

\[ |\nabla p(x) - \nabla p(y)| \leq C|x - y|^{2\alpha-1}[u]_{C^\alpha}^2. \]

I came up with these estimates by 2010. Since I could not find a good application for them, I did send them for publication. However, the result was cited at least in [1] and [2] as a personal communication.

The rest of the article consists of the proof of Theorem 1.1
1.1 Subtracting constants

We start by the following simple observation. Since \( \text{div}\, u = 0 \), the value of \( \partial_i \partial_j (u_i - A_i)(u_j - B_j) \) does not depend on \( A \) and \( B \) for any two constant vectors \( A \) and \( B \). In particular, for any two points \( x_1 \) and \( x_2 \), we have
\[
\partial_i \partial_j u_i(x) u_j(x) = \partial_i \partial_j (u_i(x) - u_i(x_1))(u_j(x) - u_j(x_2)). \tag{1.3}
\]

1.2 The case \( \alpha \in (0, 1/2) \).

Let \( \Phi(y) = \frac{c_n}{|y|^n} \) be the fundamental solution of the Laplace equation, i.e. \( \Delta \Phi = -\delta_0 \).

For any two points \( x_1 \) and \( x_2 \), let \( \varphi(y) = \Phi(y - x_1) - \Phi(y - x_2) \). We multiply both sides of (1.2) by \( \varphi \) and integrate by parts. We obtain
\[
p(x_2) - p(x_1) = \int p(y)\Delta \varphi(y) \, dy = \int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))D^2\varphi(y) \, dy
\]

We assume that \( u \) has an appropriate decay at infinity so that the tail of integral is integrable. Assuming \( u \in L^2 \) is sufficient. The estimates below do not depend on any norm of \( u \) except \( |u|_{C^\alpha} \).

Note that \( D^2 \varphi \) contains some singular part (delta functions) at \( p \) and \( y = x_2 \). However, we have that \( (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) \) vanishes for both \( y = x_1 \) and \( y = x_2 \), so we can ignore the singular part of \( D^2 \varphi \).

Let us compute the second derivatives of \( \varphi \). We start by computing \( D^2 \Phi \). We have \( D^2 \varphi(y) = D^2 \Phi(y - x_1) - D^2 \Phi(y - x_2) \), where
\[
\partial_{ij} \Phi(y) = \frac{|y|^2 \delta_{ij} - 2y_i y_j}{|y|^{n+2}}.
\]

In particular \( |D^2 \Phi(y)| \leq C|y|^{-n} \).

There is some cancellation between the two terms when \( y \) is far from \( x_1 \) and \( x_2 \). Let \( \bar{x} = \frac{x_1 + x_2}{2} \) and \( r = |x_1 + x_2| \). Then if \( |y - \bar{x}| > 5r \), by mean value theorem we have
\[
|D^2 \varphi(y)| \leq \frac{Cr}{|y - x|^{n+1}}.
\]

Therefore, we can estimate that part of the integral
\[
\int_{B_{\bar{x}}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))\partial_{ij} \varphi(y) \, dy \leq
\]
\[
\leq \|u\|_{C^\alpha} \int_{B_{\bar{x}}(\bar{x})} |y - x_1|^\alpha |y - x_2|^\alpha \frac{Cr}{|y - x|^{n+1}} \, dy
\]
\[
\leq \|u\|_{C^\alpha} \int_{B_{\bar{x}}(\bar{x})} |y - x|^{n+1-2\alpha} \, dy \leq C[u]^2 C^\alpha r^{2\alpha}
\]

Now we estimate the part of the integral where \( y \) is close to \( \bar{x} \).
\[
\int_{B_{\bar{x}}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))\partial_{ij} \varphi(y) \, dy \leq
\]
\[
\leq \int_{B_{\bar{x}}(\bar{x})} |u_i(y) - u_i(x_1)||u_j(y) - u_j(x_2)|(|D^2 \Phi(y - x_1)| + |D^2 \Phi(y - x_2)|) \, dy
\]
Note that we bound both terms, from $|D^2\Phi(y-x_1)|$ and $|D^2\Phi(y-x_2)|$, in the same way. Let us bound the first term. We use that $|u_j(y) - u_j(x_2)| \leq C[u]_{C^{\alpha}} r^\alpha$ in $B_{5r}(\bar{x})$.

$$\leq C r^\alpha \|u\|_{C^{\alpha}} \int_{B_{5r}(\bar{x})} (u_i(y) - u_i(x_1))|D^2\Phi(y-x_1)| \, dy$$

$$\leq C r^\alpha \|u\|_{C^{\alpha}}^2 \int_{B_{5r}(\bar{x})} |y-x_1|^\alpha \frac{1}{|y-x_1|^n} \, dy \leq C \|u\|_{C^{\alpha}}^2 r^{2\alpha}$$

Adding the two parts of the integral together, we obtain

$$p(x_1) - p(x_2) \leq C \|u\|_{C^{\alpha}}^2 r^{2\alpha}$$

which finishes the proof of the case $\alpha \in (0,1/2)$.

### 1.3 The case $\alpha \in (1/2, 1)$

When $\alpha \in (1/2, 1)$, $2\alpha > 1$ and the estimate obtained ($p \in C^{2\alpha}$) is actually a Hölder continuity result for $\nabla p$. The proof is slightly different because instead of estimating $p(x_1) - p(x_2)$ we have to estimate $|\nabla p(x_1) - \nabla p(x_2)|$. For that we note that

$$\nabla p(x_k) = \int (u_i(y) - u_i(x_k))(u_j(y) - u_j(x_k))\nabla \partial_{ij} \Phi(y-x_k) \, dy$$

The kernel $\nabla \partial_{ij} \Phi(y-x_k)$ has a singularity of the form $|y-x_k|^{-n-1}$ and some singular part at $y = x_k$ of order one (derivatives of Dirac delta functions). However, note that $|(u_i(y) - u_i(x_k))(u_j(y) - u_j(x_k))| \leq C|y-x_k|^{2\alpha}$ and $2\alpha > 1$, therefore the singular part of $\nabla \partial_{ij} \Phi(y-x_k)$ can be ignored and the integral above is convergent.

We write $|\nabla p(x_1) - \nabla p(x_2)|$ in integral form and divide the integral as above in the domains $|y-\bar{x}| < 5r$ and $|y-\bar{x}| \geq 5r$. Let us start with the first of these integrals.

$$\int_{B_{5r}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_1))\nabla \partial_{ij} \Phi(y-x_1) - (u_i(y) - u_i(x_2))(u_j(y) - u_j(x_2))\nabla \partial_{ij} \Phi(y-x_2) \, dy \leq$$

$$\leq 2 \left| \int_{B_{5r}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_1))\nabla \partial_{ij} \Phi(y-x_1) \, dy \right| \leq C[u]_{C^{\alpha}}^2 \int_{B_{5r}(\bar{x})} |y-x_1|^{2\alpha} \frac{1}{|y-x_1|^{n+1}} \, dy \leq C[u]_{C^{\alpha}}^2 r^{2\alpha-1}$$

Now we analyze the part of the integral where $y$ is far from $\bar{x}$.

$$\int_{B_{5r}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_1))\nabla \partial_{ij} \Phi(y-x_1) - (u_i(y) - u_i(x_2))(u_j(y) - u_j(x_2))\nabla \partial_{ij} \Phi(y-x_2) \, dy \leq$$

$$\leq \left| \int_{B_{5r}(\bar{x})} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_1))\nabla \partial_{ij} \Phi(y-x_1) - \nabla \partial_{ij} \Phi(y-x_2) \right|$$

$$+ \left| \int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_1)) - (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) \nabla \partial_{ij} \Phi(y-x_2) \, dy \right|$$

$$+ \left| \int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) - (u_i(y) - u_i(x_2))(u_j(y) - u_j(x_2)) \nabla \partial_{ij} \Phi(y-x_2) \, dy \right|$$

$$\leq C[u]_{C^{\alpha}}^2 \int_{B_{5r}(\bar{x})} |y-\bar{x}|^{2\alpha} \frac{r}{|y-\bar{x}|^{n+2}} + r^\alpha|y-\bar{x}|^\alpha \frac{1}{|y-\bar{x}|^{n+1}} \, dy$$

$$\leq C[u]_{C^{\alpha}}^2 r^{2\alpha-1}$$
Adding the two parts of the integral together, we obtain

$$|\nabla p(x_1) - \nabla p(x_2)| \leq C[u]_{C^{\alpha}}^2 r^{2\alpha - 1}$$

which finishes the proof of the case $\alpha \in (1/2, 1)$.

References
