Due 1pm on January 20th. Only problems with a * will be graded, but please submit solutions to everything.

(1) (*) From the ring axioms show that, if \(a, b \in \mathbb{Z}\), then \((-a) \times (-b) = a \times b\).

(2) (*) From the ring axioms together with cancellation show that, if \(a, b \in \mathbb{Z}\) and \(a^2 = b^2\), then \(a = b\) or \(a = -b\).

(3) Review question 6 from the last homework, in which, for each natural number \(n\), we defined the equivalence relation \(\equiv_n\) on \(\mathbb{Z}\). Let \(\mathbb{Z}_n\) be the set of equivalence classes in \(\mathbb{Z}\) under \(\equiv_n\), and write \(\bar{a}\) for the equivalence class of the integer \(a\).

(a) Explain why (6c) and (6d) from last time allow us to define operations + and \(\times\) on \(\mathbb{Z}_n\) by the formulas \(\bar{a} + \bar{b} = a + b\) and \(\bar{a} \times \bar{b} = a \times b\).

(b) (*) With these operations, \(\mathbb{Z}_n\) satisfies the ring axioms. Prove it.

(c) (*) Does \(\mathbb{Z}_6\) satisfy cancellation? What about \(\mathbb{Z}_5\) (hint: find multiplicative inverses)?

(d) Why does part (c) mean that cancellation cannot be deduced from the ring axioms?

(4) Complete the missing proof from lectures that multiplication is well-defined on \(\mathbb{Q}\).

(5) Show from the order axioms that (for \(a, b\) in an ordered ring):

(a) (*) If \(a < 0\) and \(b < 0\) then \(ab > 0\);

(b) \(a^2 \geq 0\) and if \(a^2 = 0\) then \(a = 0\);

(c) (*) If \(ab = 0\) then \(a = 0\) or \(b = 0\) (i.e. that cancellation holds in an ordered ring);

(d) \(2ab \leq a^2 + b^2\).

You may assume standard facts about rings, but must work directly from O1 - O4 (though you should feel to prove any facts you need as lemmas!).

(6) (*) Using the well-ordering principle, prove that, if \(n \in \mathbb{N}\), then

\[
\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.
\]

(7) Show that if \(a, b\) are non-zero elements of an ordered field, which are either both positive or both negative, and \(a > b\), then \(a^{-1} < b^{-1}\).

(8) (*) Show that, in an ordered field, there cannot be a smallest positive element (that is, if \(a > 0\), then there exists \(b > 0\) with \(a > b\)).

(9) (optional, harder) Find a set \(R\) with operations +, \(\times\), satisfying all of the ring axioms except MC, in which multiplication is not commutative.