The final will cover essentially everything we’ve done in the course, namely Chapter 10.1-3, Chapter 12.1-5, Chapter 13.1-2, Chapter 14 and Sections 1,2,3,6 and 9 from Chapter 15. In addition, you will certainly need to know the material from the first year calculus sequence.

Try to focus on understanding the concepts over just memorizing formulas. If you’ve memorized a formula, but don’t really understand where it came from, then you haven’t really learned it. This may seem like an unimportant distinction, but it can make a big difference in your performance on tests. If you don’t understand where a formula comes from, you won’t have any chance of figuring it out if (or when) you forget it, and you won’t be able to recognize when you have misremembered it.

The only way to really learn this stuff is by doing problems. Your textbook has plenty of examples and exercises (many, many more than have been on the homework sets) - try doing some more of them if you feel you are having trouble with any of the material.

Make sure to watch out for common mistakes:

- Make absolutely sure you know how to compute derivatives of single variable functions. Make sure you don’t get confused about things like the chain rule, or trig functions, or logarithms.
- Also make sure you can correctly compute integrals of simple functions. Make sure not to get confused by u-substitution (remember that the bounds change!).
- The fact that \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \) does not mean that \( \frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2} \); the chain rule simply doesn’t work that way.
- Don’t get the equation of a line mixed up with the equation of a plane. If I ask for the (Cartesian) equation of a line, you should have two equations. If I ask for the (Cartesian) equation of a plane, you should have one. Also, these equations should be linear.
- Also don’t get Cartesian equations confused with parametric equations
- Remember that two vectors can still be the same without having the same start and end points.
- Remember that the dot product of two vectors should not be a vector, whereas their cross product should be a vector. In general, don’t get the dot product and cross product mixed up.
- When finding the domain of a function, pay attention to what happens along the boundaries. What’s the difference between the domains of \( \sqrt{x+y} \) and \( \ln(x+y) \)?
- When evaluating multi-variable limits, remember that it is not enough to consider what happens along some path (e.g. by setting \( x = 0 \), or \( x = y \) or \( x = y^2 \) etc.). If you can find two paths that give different limits, then the limit doesn’t exist. But if you get the same limit along every path you try, you still need to do something else to show the limit exists.
- Also, when using the pinching theorem, make sure that the inequalities you are using actually work. For \( x \) and \( y \) close to 0, something like \( \frac{1}{x^4 + 3y^4} \) will be huge, so you can’t say \( \frac{x^4 \sin^2 y}{x^4 + 3y^4} \leq x^4 \sin^2 y \).
- Don’t get mixed up about what it means for a variable to be constant when taking partial derivatives. If you are doing this correctly, then partial derivatives should be no harder than single variable derivatives. If you frequently mess these up, it’s a sign that you may not really understand partial derivatives.
- When I ask you to give the linear approximation to a function, then no matter what happens, you should end up with a linear function in the end.
- Also make sure you understand how to correctly use the multivariable chain rule. This is a fairly simple and easy to apply formula, but a lot of people seem to be messing it up on the quizzes and exams.
- When using the second derivative test, remember that a critical point being a saddle point is not the same thing as the test being inconclusive.
- Also remember that I have only taught you how to use the second derivative test for functions of two variables. If you try to use it for a three (or more) variable function, you are almost certainly doing something wrong.
- When using Lagrange multipliers, remember that you CANNOT divide by zero. An equation like \( x = x\lambda \)
does NOT imply that $\lambda = 1$.

- When computing double (or triple) integrals by iterated integration, remember that the bounds on the outer integrals can’t depend on the variables in the inner integrals. Something like $\int_0^1 \int_0^x f(x,y) \,dy\,dx$ would be fine, but $\int_0^x \int_0^1 f(x,y) \,dx\,dy$ would not.
- Also make sure you are actually integrating over the correct region. Draw a picture if you need to. In particular, if your region is not a rectangle, the bounds on your integral should not all be constant.

Make sure you are comfortable with the following:

Understand the material from Math 150s/130s. In particular:

- Understand what a derivative is (i.e. as the limit of a difference quotient) and understand what it represents
  - In particular, make sure you understand why $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$, and why I call this a linear approximation to $f$.
- Make sure you know how to compute derivatives. In 151/131, you should have learnt how to compute the derivative of any function you could conceivably write down. Or you don’t remember how to do that (or don’t understand how the 151/131 material gives you that) then make absolutely sure you review that.
  - Pay special attention to the chain rule - it is the most common thing for students to get wrong.
- Understand what an integral is, and what it represents geometrically.
  - Pay attention to what type of object this is. Remember that a definite integral $\int_a^b f(x)\,dx$ is just a number, your final answer should not depend on $x$. In fact, here $x$ is a ‘dummy’ variable, it shouldn’t exist outside of the integral at all.
  - On the other hand, an indefinite integral, $\int f(x)\,dx$ is a function of $x$, and so should depend on $x$ (and should also have a ‘$+C$’).
- Know how to compute integrals using the fundamental theorem of calculus, $u$-substitution, or integration by parts. And know how to recognize which one should be used.
- Remember how $u$-substitution works. In particular REMEMBER THAT THE BOUNDS CHANGE!!!!
  - Also make sure you understand why a computation like
    $$\int_0^{\pi/2} (\sin x)^2 \cos x \,dx = \int_0^{\pi/2} u^2 \,du = \left[ \frac{u^3}{3} \right]_0^{\pi/2} = \left[ \frac{\sin^3 x}{3} \right]_0^{\pi/2} = \frac{1}{3}$$
    is wrong, even though it gives the right answer.
  - Remember, when you are computing a definite integral, your answer is just a number, and so its not necessary to substitute $x$ back in before getting the answer.
- Understand how to graph functions. If you were given a function, $y = f(x)$, how much can you figure out about the graph of $f$? What things should you pay attention to? How would you go about actually drawing a graph?
- Make sure you are familiar with the basic trig functions, sin, cos and tan. What are their values at specific points, like $\pi/3$ or $\pi/2$? What identities do they satisfy? What are their derivatives? As you may have noticed, trig functions come up pretty naturally when talking about vectors, so it’s important to know how to work with them.

**Parametric Equations (10.1-2):**

- Know what it means to give parametric equations for a curve. What does something like $x = f(t)$, $y = g(t)$ mean? Is this specifying one curve, or two? How is this different from giving a Cartesian
equation for a curve? Is there more than one way to give a set of parametric equations for a given curve?

- Know how to draw the graph of a parametrically defined curve. What things about \( f \) and \( g \) are worth paying attention to? What if you don’t have formulas for \( f \) and \( g \), just graphs of them with respect to \( t \), can you still figure out what the graph of \( y \) with respect to \( x \) looks like?

- What is the relevance of the variable \( t \)? Does it show up in your graph? How do you find the point corresponding to a given value of \( t \)? The value of \( t \) corresponding to a given point?

- What if I gave you two different parametric curves, \( x = f(t), y = g(t) \), and \( x = j(t), y = k(t) \), both using the variable \( t \). Does it matter that I used the same variable for both? Would anything change if the second one was \( x = j(s), y = k(s) \)? How would you go about finding the intersection point between the two curves?

- Know how to give parametric equations to describe simple curves like lines or circles.

- Know how to find the area bounded by a parametric curve (or multiple curves). Again, it is usually unnecessary, it makes the problem much harder.

- Do you lose any information by giving a Cartesian equation for a curve?
  - The curves \( x = \cos t, y = \sec^{-2} t \) and \( x = e^t, y = e^{-2t} \) both have Cartesian equation \( y = x^{-2} \). Are they the same curve? Why or why not?
  - What about the curves \( x = t + 1, y = 2t + 4 \) and \( x = t^3, y = 2t^3 + 2 \)? Are they the same curve?

- Know how figure out properties of the curve without finding a Cartesian equation. For example, how would you find the \( y \)-intercepts of a parametric curve?
  - In general, the parametric equations of a curve tell you pretty much everything you could reasonably want to know about that curve. If your first instinct whenever you see a parametric equation is to try to eliminate \( t \) and turn it into a Cartesian equation, then its likely that you aren’t as comfortable with parametric equations as you should be. Usually, finding the Cartesian equation is not only unnecessary, it makes the problem much harder.

- In particular, know how to find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) as functions of \( t \). Why is it okay to give them as functions of \( t \), instead of \( x \)? What would you do if I asked you for their value at a specific point \((x, y)\)?
  - Remember that, even though \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2} \). The chain rule simply doesn’t work that way!

- Know how to find the area bounded by a parametric curve (or multiple curves). Again, it is usually not necessary to find Cartesian equations for your curves - if you try to use Cartesian equations, the integrals will often be much more complicated.
  - If you are having trouble figuring out how to set up the integral, it may help to draw a picture.

**Polar Coordinates (10.3):**

- Understand how the polar coordinates \((r, \theta)\) describe a point in the \(xy\)-plane. For Cartesian coordinates, you specify two pieces of information (the distances from each of the axes) to describe a point. What two pieces of information are you giving here? Why do they tell you a specific point?
  - In Cartesian coordinates, any point can be described by exactly one pair of coordinates. Is that true here? Can a single point be described in more than one way using polar coordinates? How many ways?
  - What does it mean when \( r < 0 \)? When \( r = 0 \)? When \( \theta < 0 \)? When \( \theta > 2\pi \)?

- Know how to convert a point from polar coordinates to Cartesian coordinates, and vice versa. Make sure you know how to express \( x \) and \( y \) in terms of \( r \) and \( \theta \), and vice versa.

- Understand what a polar equation like \( r = f(\theta) \) represents. What does it mean to graph something like this? Is this the same thing as graphing \( y = f(x) \)?
• Know how to graph a polar equation like \( r = f(\theta) \). What do various properties of \( f \) mean for the graph? What happens if \( f(\theta + 2\pi) = f(\theta) \)? If \( f(\theta + 2\pi) \neq f(\theta) \)? If \( f(\theta) \) is increasing? If \( f(\theta) \) is negative for some values of \( \theta \)? If \( f(\theta) \) is constant? If \( f(\theta + \pi/2) = f(\theta) \)?

• Know how to convert a polar equation into a Cartesian equation and vice versa. To turn a Cartesian equation into a polar equation, it’s usually enough to just substitute in \( x = r \cos \theta \), \( y = r \sin \theta \). The other direction can sometimes be a little harder, but it’s still just a matter of using the equations relating \( x \) and \( y \) to \( r \) and \( \theta \).

• Know how to find the slope of a tangent line to \( r = f(\theta) \) without turning the equation into a Cartesian equation. This is pretty similar to parametric equations. You can simply write \( \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \), and use the equations \( x = r \cos \theta \), \( y = r \sin \theta \) and \( r = f(\theta) \). Remembering this procedure is probably much more useful than trying to remember the actual formula.

THREE DIMENSIONAL COORDINATES (12.1):

• Know how use three coordinates \((x, y, z)\) to describe a point in three dimensional space.
  – In two dimensions, you need to pieces of information to describe a point (which can either be its Cartesian coordinates or its polar coordinates), in three dimensions, you need three.

• In what ways is this similar to two-dimensional Cartesian coordinates? What (if anything) is different.

• Know what is meant by: the origin, the \( x, y \) and \( z \) axes, the \( xy, yz \) and \( xz \) planes, and the octants. How can you tell if a point \((x, y, z)\) lies on one of these things? What does it mean in terms of the \( x, y \) and \( z \) coordinates?

• In two dimensions, giving a single equation will (usually) describe a curve. What would a single equation describe in three dimensions? Would it also be a curve? How would you typically describe a curve? How many equations would you need?

• What if you give a set of parametric equations \( x = f(t), y = g(t), z = h(t) \)? What type of object does that describe? A curve? A surface? Something else?

• In two dimensions, an equation like \( y = f(x) \) describes a curve. If you take the same equation in three dimensions, it will now give you a surface. How does that surface relate to the original curve?

• Know how to use the distance formula to find the distance between two points in three dimensions.

• Know how to write the equation of a sphere with a given center and radius. If you were given some equation, how would you figure out if it was the equation of a sphere? How would you find the center and radius?

• If \( f(x, y, z) = C \) describes a surface, what type of object would an inequality like \( f(x, y, z) < C \) or \( f(x, y, z) > C \) describe? If you are given some sort of solid object in three dimensions, know how to write down inequalities describing it.

VECTORS (12.2):

• Understand what a vector is - it is a object with a length and a direction. What are some real life examples of these? How is this different from a scalar?

• Make sure you understand exactly what information is contained in a vector? Does it matter where the starting point of the vector is? Is there any difference between the vectors \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) below?
• What is the difference between a point and a vector? Is there any?
• Understand geometrically what it means to add two vectors, or to take the scalar multiple of a vector. If $c < 0$, then what does $ca$ mean?
• Remember, two vectors have the same direction if and only if one is a scalar multiple of the other.
• Know how to express a vector in terms of components. If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, how can you express the vector $\overrightarrow{AB}$?

Now if $v = \langle a, b, c \rangle$ and $v$ starts at $A = (x_1, y_1, z_1)$. Where is the endpoint of $v$?

• How do you use the components of a vector to calculate $a + b$? $ca$? What about the length of $a$?
• Know what a unit vector is. How do you find a unit vector with the same direction as these?

Know what a unit vector is. How do you find a unit vector with the same direction as $a$?

• Know what the cross product of two vectors is, and understand why it is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. What is the difference between a point and a vector? Is there any?
• Remember, two vectors have the same direction if and only if one is a scalar multiple of the other.
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• Know what a unit vector is. How do you find a unit vector with the same direction as $a$?

Understand what a vector valued function is, that is, a function which takes real numbers and outputs vectors.

Understand why this describes a curve. How does this relate to giving parametric equations for a curve?

Understand why a vector valued function $\mathbf{r}(t) = (f(t), g(t), h(t))$ is just the same as giving 3 functions $f(t)$, $g(t)$ and $h(t)$. In terms of $f, g$ and $h$, how do you tell if $\mathbf{r}$ is continuous? How do you take a limit of $\mathbf{r}$? What is the domain of $\mathbf{r}$? How do you find the derivative of $\mathbf{r}$? The integral? Once you understand what’s going on, these questions are no harder than for real valued functions.

Know how to describe the equation of a line passing through a point $A$, parallel to a vector $\mathbf{v}$. What about the line passing through two points, $A$ and $B$?

Know how to visualize what the graph of a vector valued function looks like. It may help to find Cartesian equations that $f(t)$, $g(t)$ and $h(t)$ satisfy. For instance, if $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, then any point on $\mathbf{r}$ satisfies $y = x^2$ and $z = x^3$. What does this mean geometrically?

Know how to find the tangent vector to a curve at a specific point. What about the unit tangent vector?

How can you find an equation for the line tangent to a curve at a given point?

Know how to differentiate combinations of vector valued functions like $\mathbf{u}(t) + \mathbf{v}(t)$, $f(t)\mathbf{u}(t)$ of $\mathbf{u}(f(t))$. These should be almost exactly to same as for single variable functions, just remember that the product rule holds for all three types of products of vectors (scalar multiplication, dot product and cross product).

Dot Products (12.3):

• Know what the dot product is, and know how to use it find the angle between two vectors.
  – The most natural way to find the angle between two vectors is to use the law of cosines. Understand why this simplifies to $\mathbf{a} \cdot \mathbf{b} = |a||b| \cos \theta$.

Know the basic properties of dot products, such as $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ or $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. Know how to show these by looking at components.

Remember that $\mathbf{a} \cdot \mathbf{b}$ is a scalar (i.e. a number) not another vector.

Know what the quantity $\mathbf{a} \cdot \mathbf{a}$ means geometrically.

What $\mathbf{a} \cdot \mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (i.e. perpendicular)? What about when they are parallel?

Know how to use dot products to find the angle between objects other than vectors (such as lines, or graphs). Your first step should be to describe the angle you want in terms of vectors (how do you do this?).

Understand the scalar projection, $\text{comp}_\mathbf{a} \mathbf{b}$ and vector projection $\text{proj}_\mathbf{a} \mathbf{b}$ of $\mathbf{b}$ onto $\mathbf{a}$. How do you compute these? What do they mean geometrically? What can they be used for?

What is the scalar projection of $\mathbf{a}$ onto $\mathbf{i}$ (or $\mathbf{j}$ or $\mathbf{k}$)? How does this relate to the components of $\mathbf{a}$?

Cross Products (12.4):

• Know what the cross product of two vectors is, and understand why it is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. Why did we come up with the definition we did?
• We call the cross product a product, what common properties of a product does it satisfy? Does 
\((a + b) \times c = a \times c + b \times c\)? Does \(a \times b = b \times a\)? Does \((a \times b) \times c = a \times (b \times c)\)?

• Notice that the cross product is a different sort of object than the dot product - \(\mathbf{a} \cdot \mathbf{b}\) is a scalar, not a vector, whereas \(\mathbf{a} \times \mathbf{b}\) is still a vector.

• Also remember that, unlike any of the other vector concepts we’ve talked about, the cross product makes sense only in three dimensions. You can’t take the cross product of \(n\)-dimensional vectors.

• Know what the vector \(\mathbf{a} \times \mathbf{a}\) is.

• If you know about determinants, try to learn the \(3 \times 3\) determinant formula for the cross product. If you don’t know about determinants, don’t worry about this.

• Know how to take the cross products of \(\mathbf{i}, \mathbf{j}\) and \(\mathbf{k}\). Remember \(\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}\), not \(\mathbf{j}\). Instead, \(\mathbf{k} \times \mathbf{i} = \mathbf{j}\).

• If you don’t think that you can correctly memorize the formula for the cross product, don’t worry about it. Know how to find the cross product of any two vectors by expressing them in terms of \(\mathbf{i}, \mathbf{j}\) and \(\mathbf{k}\), and distributing. This may take a bit longer than using the formula, but you are much less likely to make a mistake.

• We know that \(\mathbf{a} \times \mathbf{b}\) is orthogonal to \(\mathbf{a}\) and \(\mathbf{b}\), but there are still two possible directions. Know how to use the right hand rule to find the direction of \(\mathbf{a} \times \mathbf{b}\).

• Know how to express \(|\mathbf{a} \times \mathbf{b}|\) in terms of |\(\mathbf{a}\)|, |\(\mathbf{b}\)| and the angle \(\theta\) between \(\mathbf{a}\) and \(\mathbf{b}\). What does this mean geometrically?

• Know how to use cross products to find the area of a parallelogram. What about a triangle?

• Know how to use cross products to determine if two vectors are parallel.

• Know how to use the triple product to find the volume of a parallelepiped, and to figure out if three vectors are coplanar.

Equations of Lines and Planes (12.5):

• Know how to find the Cartesian equations for a line through a point \(A\), parallel to a vector \(\mathbf{v}\). What about the line through two points \(A\) and \(B\)?

• Know what it means for two lines to be parallel. What about to be skew?

• Know how to use cross products to find a normal vector to a plane. Know how to use the normal vector to a plane to find a Cartesian equation for a plane. You should be able to use this to find the equation of a plane through three points. (What if the cross product you try to compute gives you the zero vector, what does that mean about the points you picked?)

• Know how to find the intersection point of a line and a plane when: the line is given by Cartesian equations; the line is given by parametric equations.

• Know how to find the equation (parametric or Cartesian) for intersection of two planes.

• Know how to use normal vectors to find the angle between two planes. When are the planes perpendicular? Parallel?

• Know how to find the distance from a point to a plane. What about the distance between two parallel planes? The distance between two parallel lines? Two skew lines?

Functions of Multiple Variables: (Section 14.1)

• Know what it means for \(f\) to be a function of two variable. What about 3? Or \(n\)? What does it mean to say \(z = f(x, y)\)? (or \(w = f(x, y, z)\)? or \(y = f(x_1, \ldots, x_n)\)?) What does it mean to say the domain of such a function is \(D\)? What is the range?

• Know how to find the domain of a function like \(f(x, y) = x \ln(y^2 - x)\) or \(f(x, y, z) = \sqrt{x + y + z}\). Pay attention to what happens at the boundaries!

• Know what it means to graph a multivariable function. How many dimensions do you need to draw the graph of \(f(x, y)\)? What about \(f(x, y, z)\)?

• Know what a level curve (or level surface) of a function is. Know how to use these to draw the contour map of a function. Also know how to interpret a contour map. What do maxima/minima look like?
How do you tell if the value of \( f \) is changing quickly or slowly?

**LIMITS AND CONTINUITY: (SECTION 14.2)**

- Know what \( \lim_{(x,y) \to (a,b)} f(x, y) = L \) means. What does it mean intuitively? What is the formal definition (involving \( \varepsilon \) and \( \delta \))? Why does that definition match your intuition? What happens with more than two variables?
- What needs to be true for \( \lim_{(x,y) \to (a,b)} f(x, y) \) to exist. Is it enough to express things in terms of one variable limits?
- Understand the difference between the limits \( \lim_{(x,y) \to (a,b)} f(x, y) \) and \( \lim_{x \to a} \lim_{y \to b} f(x, y) \).
  - Is it possible for the second one to exist, but not the first one? The first one but not the second one? What’s missing in the second limit? Why doesn’t it tell us everything about the first limit?
  - If the two limits both exist, must they be equal?
- Is it enough to look at \((x, y) \to (a, b)\) along the both the lines \( x = a \) and \( y = b \). What about along every line through \((a, b)\).
- Understand what it means to say that \( \lim_{(x,y) \to (a,b)} f(x, y) = L \) means that \( f(x, y) \to L \) along every path to \((a, b)\). Know how to use this to show that a limit does not exist. Can you use this to show that a limit does exist?
- Know what is means for a function to be continuous. How do you recognize continuous functions. How can you use continuity to help evaluate limits?
- Know how to use properties of limits, such as the pinching theorem to evaluate limits where the functions are discontinuous, or not defined (such as \( \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} \)).
- What, if anything, is different if you talk about limits in three or more variables?

**PARTIAL DERIVATIVES: (SECTION 14.3)**

- Know what it means to take a partial derivative of a multivariable function with respect to one of a variables.
- Be familiar with the common notations for this: \( \frac{\partial z}{\partial x}, \frac{\partial}{\partial x} f(x, y), f_x(x, y) \) etc.
- Know how to compute these. Taking \( \frac{\partial}{\partial x} f(x, y) \) just means taking a one variable derivative, where \( y \) (and any variables besides \( x \) are treated as constant). Make absolutely sure you fully understand what this means?
  - How would you handle an expression like \( \frac{\partial}{\partial x} \sin(xy) \)? What does it mean that \( y \) is constant here?
  - What about something like \( \frac{\partial}{\partial x} \left[ \frac{y^{1/2} + e^{\tan y}}{y^{e^{xy}} + \sqrt{y^3 - \ln y}} \right] \)? Do you need to do a lot of work to compute this?
  - If you understand how all of this works, then partial derivatives should be no harder for you than single variable derivatives.
- Know how to interpret partial derivatives in terms of slopes, or as rates of change.
- Know what a second (or higher) partial derivative of a function is. In one variable, there’s only one thing we can call the second derivative of a function \( f(x) \). How many different second derivatives of a function \( f(x, y) \) are there? What’s the difference between them?
- Know how to compute these. Again, if you understand what’s going on, these should be no harder than single variable derivatives.
- Remember that an expression like \( \frac{\partial^2}{\partial x \partial y} f(x, y) \) means that you first treat \( x \) as a constant and \( y \) as a variable, and then you treat \( y \) as a constant as \( x \) as a variable. Don’t get mixed up.
• Understand why \( \frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y) \) for almost every function \( f \). Know how to take advantage of this to make your life easier.

• What, if anything, is different if you talk about partial derivatives in three or more variables?

**Directional Derivatives: (Section 14.6)**

• Understand why the partial derivatives of a function may not tell you everything you want to know about how the function changes - they only tells you what happens when you move \((x,y)\) parallel to the \(x-\) or \(y-\) axes.

• Know how to define a directional derivative, \( D_u f(x, y) \), and what this means conceptually. \( D_u f \) is telling you how \( f \) changes when your point moves in the \( u \) direction.
  
  – Keep in mind that your textbook requires \( u \) to only be a unit vector, but I have made no such restrictions.
  
  – Understand why \( D_{au} f = a D_u f \), and why it means there’s nothing really different about allowing non-unit vectors. All it does is make your computations slightly simpler.

• Understand how to compute these directly from the definition.

• Know how these relate to the partial derivatives. What are \( D_1 f \) and \( D_2 f \)?

• Again, what changes (if anything) when you have more than two variables?

• Understand why the directional derivatives still aren’t a completely satisfactory description of the derivative of \( f \). Knowing all of the directional derivatives does tell you how \( f \) changes, but that requires know infinitely many pieces of information!

**Tangent Planes and Linear approximation: (Section 14.4)**

• Understand the formula \( f(x + h) \approx f(x) + f'(x)h \) in one variable, and why I have called this the linear approximation to \( f \). Also understand why giving a linear approximation to a function is the same as giving a tangent line, and so the derivative (i.e. the slope of the tangent line) is basically the “same thing” as the linear approximation to the function.

• Understand the definition of a linear approximation. Why do I say that \( f(x + h) = f(x) + f'(x)h + e(x)h \), where \( e(x) \to 0 \) as \( h \to 0 \) is the same thing as saying \( f(x + h) \approx f(x) + f'(x)h \), and why this can be used as an alternate definition of \( f \) being differentiable.

• Understand how to generalize this to two (or more!) variables.
  
  – We say that \( f \) is differentiable at \((x_0, y_0)\) if

  \[
  f(x, y) = f(x_0, y_0) + m(x - x_0) + n(y - y_0) + e_1(x, y)(x - x_0) + e_2(x, y)(y - y_0)
  \]

  for some \( m, n \) where \( e_1, e_2 \to 0 \) as \((x,y) \to (x_0, y_0)\), and we say that the linear function \( L(x, y) = f(x_0, y_0) + m(x - x_0) + n(y - y_0) \) is the linear approximation (or equivalently, \( z = L(x, y) \) is the tangent plane) to \( f \) as \((x_0, y_0)\).

  – Understand why this definition makes sense. Why does it imply \( f(x, y) \approx L(x, y) \) for \((x, y) \approx (x_0, y_0)\)?

• Know how to compute \( D_u f \) in terms of \( m \) and \( n \), provided \( f \) is a differentiable function. Know why this implies that \( m = f_x(x_0, y_0) \) and \( n = f_y(x_0, y_0) \), and therefore that

\[
 f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

• Know how to use this to find the approximate value of \( f(x, y) \) near some point, or to find the amount by which \( f \) changes when \( x \) and \( y \) both change slightly.

• Know how to tell if a function is differentiable. Remember that its not enough to know that \( f_x \) and \( f_y \) both exist at \((x_0, y_0)\), but it is enough to know that they also exist, and are continuous near \((x_0, y_0)\). Know why this implies that almost every function you will encounter is differentiable, just like in one variable.

• Again, does anything change with more than two variables?
Gradient Vectors: (Section 14.6)

- Understand why the formula $D_{(a,b)}f = af_x + bf_y$ can be expressed as $D_u f = \nabla f \cdot u$, where $\nabla f = (f_x, f_y)$ is the gradient vector. Understand why this means that just knowing $\nabla f(x_0, y_0)$ is (usually) enough to know how $f$ changes near $(x_0, y_0)$.
- What does $\nabla f$ mean in terms of the function $f$? In which direction does $f$ increase the most rapidly (i.e. when is $D_u f$ maximized, for $u$ a unit vector). What is this maximum rate of change?
- Understand how the formula above can be re-written as $D_u f = |\nabla f| \cos \theta f_u$, and why this matters. This is essentially saying that not all of the vector $u$ matters when determining how $f$ changes, only its component in the $\nabla f$ direction. So essentially the value of $f$ is only affected by moving $(x, y)$ in one direction. All of the other motion can be ignored. This is why the notion of a gradient is so powerful.
- Know how the gradient of a function relates to its level curves (if moving in the direction of $\nabla f$ has the largest effect on the value of $f$, then which direction should one move in to not change the value of $f$). Know how to use this to find normal vectors and tangent lines (or planes) to curves and surfaces.
- Again, does anything change with more than two variables?

The Chain Rule: (Section 14.5):

- Understand how, in one variable, the approximations $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$ and $f(y + \Delta y) \approx f(y) + f'(y)\Delta y$ can be used to determine $\frac{d}{dx}g(f(x))$.
- Know how to use the linear approximation to a differentiable function $f(x, y)$ to find the derivative $\frac{d}{dt}f(g(t), h(t))$ without explicitly computing the function $p(t) = f(g(t), h(t))$.
- Understand what the formula $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$ means. In particular, how do you compute the derivative at the point $t = t_0$? What values of $x$ and $y$ do you plug into $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ or $\frac{dz}{dt}$. What if we have more variables?

Maximum and Minimum Values: (Section 14.7)

- Understand why $(x_0, y_0)$ cannot be a local maximum or minimum value for $f$ if $\nabla f(x_0, y_0) \neq 0$.
- Know how to use this to find the possible local maxima and minima for a function. These can only occur at the critical points of $f$, namely points where $\nabla f = 0$ or $f$ fails to be differentiable. Understand how to find the critical points of a given function $f$ (this will involve solving the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$).
- Know what a saddle point of a function is (i.e. a point $(x_0, y_0)$ which is a minimum along one line through $(x_0, y_0)$ and maximum along a different line). Know what this looks like geometrically (think about the graph of $z = y^2 - x^2$) and understand why this can't happen in one dimension.
- Know how to use the second derivative test to figure out if a critical point is a local maximum, a local minimum or a saddle point.
  - Make sure you get this formula right, it may be more confusing than most of the formulas in this course, and I unfortunately can't give you a satisfactory explanation for it without using linear algebra. At least make sure you understand why it works in the case where $f_{xy} = 0$. This can help you figure out what's correct if you can't quite remember the formula.
  - Remember that $D > 0$ at both minima and maxima. If $D > 0$, how do you tell if your point is a maximum or a minimum? What does it mean if $D < 0$?
  - When is the test inconclusive? Is this different from saying that your point is a saddle point?
As a warning, unlike most things in this course, I have not told you how to use the second derivative test with three or more variables (critical points will work the same). To do so would require material from Math 196 (specifically, eigenvalues), and so it will not be covered in this course.

Know how to use the extreme value theorem to find the absolute maxima and minima of a function on a closed and bounded domain. Remember that this does not require the second derivative test (why not?) but does involve finding the maxima and minima of the function along the boundary of the domain.

**Lagrange Multipliers: (Section 14.8)**

- Understand why finding the points where \( \nabla f = 0 \) is not sufficient to find the maxima and minima of a function \( f \) along a curve \( g(x, y) = k \).
  - Make absolutely sure you can recognize when you are in a situation like this. Knowing how to use Lagrange multipliers will be useless to you if you don’t know when you need to use them.

- Understand why a point \((x_0, y_0)\) on \( g(x, y) = k \) cannot be a minimum or maximum if \( \nabla f \) and \( \nabla g \) are not parallel.

- If \( \nabla g \neq 0 \), understand why this condition can be expressed as \( \nabla f = \lambda \nabla g \). (What’s wrong with writing it as \( \nabla g = \lambda \nabla f \)?)

- Know how to use this to find the minimum and maximum values of \( f(x, y) \) on the curve \( f(x, y) = k \). Note that this will involve solving for three variables \( x, y \) and \( \lambda \), so you will need three equations? How many equations are contained in the equality \( \nabla f = \lambda \nabla g \). Where can you get another one.

- Make sure you know how to solve these equations.
  - Do NOT neglect this fact when studying for the exam. It is extremely easy to look over a few examples and convince yourself that you know how to do this, and then realize on the test that you have no clue what to do after writing down the equations.

  - These equations are non-linear, so this is not simply a matter of memorizing a few steps and applying that to any problem you encounter. You need to think of each one as its own problem, and need to come up with a new approach each time.

  - Remember basic tricks for solving equations. Try combining some equations to eliminate a variable, or otherwise simplify things. Try solving for some variables in terms of another variable (say, express \( x \) and \( y \) in terms of \( \lambda \)) and then substitute that into another equation.

  - Remember that you can only divide by things that are nonzero. That means whenever you are dividing equations, you need to check that some variables are nonzero. In some cases, there may be maxima/minima at points where a variable is zero, which you would miss if you just divided without checking this.

  - The ONLY way to really get good at this is by doing problems. If HW 7 wasn’t enough practice, do some more exercises from the textbook.

- Also know how to deal with a situation where you have more than one constraint.

- Again, what changes when you have more than two variables?

**Double integrals: (Section 15.1-2)**

- Understand what a double integral \( \iint_R f(x, y) dA \), where \( R = [a, b] \times [c, d] \) is a rectangle, means both in terms of Riemann sums, and as a volume.

- Know how to express this as an iterated integral, \( \int_a^b \int_c^d f(x, y) \, dy \, dx \), and understand what this means (you are first treating \( x \) as a constant as integrating with respect to \( y \)). Make sure you know how to compute this. Just like for partial derivatives, this should be know harder than for single variables if you understand what it means that one variable is constant.

- Know how you can sometimes switch the order in which you integrate \( x \) and \( y \) to simplify the computations.

- Know what it means to integrate over a region, \( D \), that is not a rectangle.
• If $D$ is the region defined by $a \leq x \leq b$, $f(x) \leq y \leq g(x)$, know how to express $\int\int_D f(x,y)\,dA$ as an iterated integral and how to compute it (notice that now it really matters which order you integrate in). Remember that the bounds on the inner integral may not be constant. What if $x$ is between $f(y)$ and $g(y)$ instead.

• Know how to break up an integral over a more complicated region into integrals over simpler regions.

• Know what happens when you switch the order of integration over a region that isn’t a rectangle. (Pay attention to what happens to the bounds!).

• Know how to easily integrate $\int\int_D f(x)g(y)\,dA$, where $R = [a,b] \times [c,d]$ is a rectangle. Does this trick work if $D$ isn’t a rectangle?

• Know what $\int\int_D 1\,dA$ represents geometrically.

**Triple Integrals:** (Section 15.6):

• Understand what a triple integral $\int\int\int_B f(x,y,z)\,dV$, where $R = [a,b] \times [c,d] \times [e,f]$ is a rectangular box, means both in terms of Riemann sums, and geometrically (if the general geometric interpretation is too confusing, at least understand what the integral represents when $f$ is constant.)

• Also understand how this works if the region of integration is not a box.

• Know how to express a triple integral as in iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$, and how to compute this.
  
  – Pay special attention to the bounds here. Which variables are allowed as bounds in which integrals?
  
  How is this affected by the order of integration. How can you tell the difference between an iterated integral which makes sense, and one which does not?

  – Also make sure you know how to find the bounds for an iterated integral if I just describe some region $E$ to you in three dimensional space.

• Understand how to change the order of integration in three variables. Make sure you know how to figure out the new bounds. Also remember that there are six possible ways to order the variables. Don’t get them mixed up, and try to think about which order will be best for computing a given integral.

**Change of Variables:** (Section 15.9)

• Understand why $u$-substitution, $\int_a^b f(x)\,dx = \int_a^b f(g(u))g'(u)\,du$ works, without appealing to the fundamental theorem of calculus. (In particular, where does the $g'(u)$ come from? What does it represent?)

• Understand what needs to be changed when you switch from one to two variables. In particular, what is $g'(u)$ replaced by? What does this represent geometrically?

• Understand how transformations work in 2 variables. How do you figure out what a given region in the $uv$-plane maps to in the $xy$-plane? What region in the $uv$-plane maps to a given region in the $xy$-plane? Also know how to do this in three variables.

• Know how to compute Jacobians $\frac{\partial(x,y)}{\partial(u,v)}$ (or $\frac{\partial(x,y,z)}{\partial(u,v,w)}$), and understand where these formulas come from. While the formula is a bit complicated in three variables, remember that it can be simpler in a lot of specific cases.

• Know how to compute a double (or triple) integral using change of variables. If a specific variable substitution is not given to you, know how to possibly find one (either by making the function $f$ simpler, or by making the region of integration simpler). Keep in mind that there may be more than one variable substitution which may work. Try to find one that will make your life as easy as possible.

• Know how to sometimes find a variable substitution that will turn a complicated region into a rectangle, or other simple shape. It may help to write the boundaries of the region in the form $f(x,y) = a$, and
then let $u = f(x, y)$, or to find parametric equations for the boundaries.

- Know how to use change of variables to find the area (or volume) of a possibly complicated region. Note that this will often involve calculating the integral of a nonconstant function.

**Polar Coordinates: (Section 15.3)**

- Know how to use the change of variables formula to rewrite an integral in Cartesian coordinates in terms of polar coordinates, and how to evaluate this integral.
- This is essentially just using the variable transformation $x = r \cos \theta$, $y = r \sin \theta$. What is the Jacobian $\frac{\partial(x, y)}{\partial(r, \theta)}$ in this case?
- Know how to recognize regions of integration that would be simpler to describe in terms of polar coordinates.