This homework is due in class on Wednesday, October 29th. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.

1. In each of the problems below, sketch the graph of a function $f$, defined on $[0, 1]$, which satisfies the given conditions, or state that no such function exists. For this question only you do not need to give any explanation beyond an answer.

   (a) (Exercise 2.6.13) $f$ is continuous on $[0, 1]$ with minimum value 0 and maximum value $\frac{1}{2}$.
   (b) (Exercise 2.6.14) $f$ is continuous on $[0, 1)$ with minimum value 0 and no maximum value.
   (c) (Exercise 2.6.16) $f$ is continuous on $[0, 1]$, takes on the values $-1$ and 1, but does not take on the value 0.
   (d) (Exercise 2.6.22) $f$ is continuous on $(0, 1)$ and takes on only three distinct values.
   (e) (Exercise 2.6.23) $f$ is continuous on $(0, 1)$, and the range of $f$ is an unbounded interval.

2. For each of the functions below, define the function at 1 so that it becomes continuous at 1, if possible:

   (a) (Exercise 2.4.31) $f(x) = \frac{x^2 - 1}{x - 1}$.
   (b) (Exercise 2.4.32) $f(x) = \frac{1}{x - 1}$
   (c) (Exercise 2.4.33) $f(x) = \frac{x - 1}{|x - 1|}$
   (d) (Exercise 2.4.34) $f(x) = \frac{(x - 1)^2}{|x - 1|}$

3. In each case below, determine the values of $A$ which make $f(x)$ continuous on $(-\infty, \infty)$.

   (a) (Exercise 2.4.35) $f(x) = \begin{cases} x^2 & x < 1 \\ Ax - 3 & x \geq 1 \end{cases}$
   (b) (Exercise 2.4.36) $f(x) = \begin{cases} A^2x^2 & x \leq 2 \\ (1 - A)x & x > 2 \end{cases}$
4. Prove the following statements about a function $f$.

(a) If $\lim_{x \to c} f(x) = L$, then there is some $p > 0$ and some $M$, such that $|f(x)| < M$ for $x \in (c - p, c + p)$, except possibly at $x = c$.

(b) (Exercise 2.5.47) If there are real number $B$ and $p > 0$ such that $|f(x)| \leq B$ for all $x \in (-p, p)$ with $x \neq 0$, then $\lim_{x \to 0} xf(x) = 0$.

(c) (Exercise 2.4.55) If $\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$ exists, then $f$ is continuous at $c$.

5. Let $f(x)$ be the Dirichlet function:

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational}. \end{cases}$$

(a) Prove that $f(x)$ is not continuous at any point in $(-\infty, \infty)$.

(b) Prove that $g(x) = xf(x)$ is continuous at $x = 0$.

6. Use the intermediate value theorem to show that there is a solution to each of the given equations in the given intervals:

(a) (Exercise 2.6.1) $2x^3 - 4x^2 + 5x - 4 = 0$ in $[1, 2]$.

(b) (Exercise 2.6.3) $\sin x + 2\cos x - x^2 = 0$ in $[0, \pi/2]$.

(b) (Exercise 2.6.4) $2\tan x - x = 1$ in $[0, \pi/4]$.

(d) (Exercise 2.6.7) $x^3 = \sqrt{x} + 2$ in $[1, 2]$.

7. (Exercise 2.6.26) Given that $f$ and $g$ are continuous on $[a, b]$, and that $f(a) < g(a)$ and $f(b) > g(b)$, show that there exists at least one number $c \in (a, b)$ such that $f(c) = g(c)$. (Hint: Consider the function $h(x) = g(x) - f(x)$.)