1-23. If \( f: A \to \mathbb{R}^m \) and \( a \in A \), show that \( \lim_{x \to a} f(x) = b \) if and only if \( \lim_{x \to a} f_i(x) = b_i \) for \( i = 1, \ldots, m \).

**Solution.** Suppose \( \lim_{x \to a} f(x) = b \). To each \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that \( |f(x) - b| < \varepsilon \) whenever \( 0 < |x - a| < \delta \). Hence

\[
|f_i(x) - b_i| \leq |f(x) - b| < \varepsilon
\]

when \( 0 < |x - a| < \delta \). Conversely, suppose \( \lim_{x \to a} f_i(x) = b_i \). To each \( \varepsilon > 0 \), there exists \( \delta_i \) such that \( |f_i(x) - b_i| < \varepsilon / \sqrt{m} \) when \( 0 < |x - a| < \delta_i \). Put \( \delta = \min\{\delta_1, \ldots, \delta_n\} \). Then whenever \( 0 < |x - a| < \delta \),

\[
|f(x) - a| = \sqrt{\sum_{i=1}^{m} |f_i(x) - b_i|^2} = \sqrt{\sum_{i=1}^{m} \varepsilon^2 / m} = \varepsilon.
\]

1-24. Prove that \( f: A \to \mathbb{R}^m \) is continuous at \( a \) if and only if each \( f_i \) is.

**Solution.** Put \( b = f(a) \) in the previous exercise.

1-25. Prove that a linear transformation \( T: \mathbb{R}^n \to \mathbb{R}^m \) is continuous.

**Solution.** Choose \( M \) as in problem 1-10, and fix \( \varepsilon > 0 \). Then

\[
|Tx - Ty| = |T(x - y)| \leq M|x - y| < M \cdot \frac{\varepsilon}{M} = \varepsilon
\]

whenever \( |x - y| < \varepsilon / M \), so \( T \) is (uniformly) continuous.

1-27. Prove that \( \{x \in \mathbb{R}^n : |x - a| < r\} \) is open by considering the function \( f: \mathbb{R}^n \to \mathbb{R} \) with \( f(x) = |x - a| \).

**Solution.** For any \( \varepsilon > 0 \), the reverse triangle inequality implies

\[
|f(x) - f(c)| = ||x - a| - |c - a|| < |(x - a) - (c - a)| = |x - c| < \varepsilon
\]

whenever \( |x - c| < \varepsilon \), so \( f \) is continuous. Note that

\[
\{x \in \mathbb{R}^n : |x - a| < r\} = f^{-1}((0, r)).
\]

Since the continuous inverse image of open sets is open, the set is open.

1-28. If \( A \subset \mathbb{R}^n \) is not closed, show that there is a continuous function \( f: A \to \mathbb{R} \) which is unbounded.

**Solution.** The set \( \mathbb{R}^n - A \) is not open, so there is a point \( b \notin A \) such that \( B(b, \varepsilon) \cap A \neq \emptyset \) for all \( \varepsilon > 0 \). Define a function \( f: A \to \mathbb{R} \) by \( f(x) = 1/|x - b| \). To see that \( f \) is unbounded, let \( M > 0 \) be given, and pick \( x \in B(b, 1/M) \). Then \( |x - b| < 1/M \), so \( f(x) > M \).
2-1. Prove that if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is differentiable at \( a \in \mathbb{R}^n \), then it is continuous at \( a \).

Solution. Let \( T = Df(a) \) and \( M \) be number such that \( |Th| \leq M|h| \) for all \( h \). Choose \( \delta > 0 \) such that \( 0 < |h| < \delta \) implies
\[
\frac{|f(a + h) - f(a) - Th|}{|h|} < 1.
\]
For any \( \varepsilon > 0 \)
\[
|f(a + h) - f(a)| \leq |f(a + h) - f(a) - Th| + |Th| \\
\leq |h| + M|h| \\
= (1 + M)|h| \\
< \varepsilon,
\]
whenever \( |h| < \min(\delta, \varepsilon/(1 + M)) \), so \( f \) is continuous at \( a \).

2-4. Let \( g \) be a continuous real-valued function on the unit circle \( \{ x \in \mathbb{R}^2 : |x| = 1 \} \) such that \( g(0,1) = g(1,0) = 0 \) and \( g(-x) = -g(x) \). Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
|x| \cdot g \left( \frac{x}{|x|} \right) & x \neq 0 \\
0 & x = 0.
\end{cases}
\]
(a) If \( x \in \mathbb{R}^2 \) and \( h : \mathbb{R} \to \mathbb{R} \) is defined by \( h(t) = f(tx) \), show that \( h \) is differentiable.
(b) Show that \( f \) is not differentiable at \((0,0)\) unless \( g = 0 \).

Solution.
(a) Note that if \( x \neq 0 \) and \( t \neq 0 \),
\[
f(tx) = |tx| \cdot g \left( \frac{tx}{|tx|} \right) \\
= |t||x|g \left( \frac{tx}{|t||x|} \right) \\
= |t||x|g \left( sgn(t) \frac{x}{|x|} \right) \\
= sgn(t)|t||x|g \left( \frac{x}{|x|} \right) \\
= tf(x),
\]
and if either \( t = 0 \) or \( x = 0 \), we have \( tx = 0 \) so that \( f(tx) = 0 \). Hence \( h(t) = tf(x) \), so \( h \) is differentiable.

(b) First note that since \( g \) is odd, we have \( g(-1,0) = g(-(1,0)) = -g(1,0) = 0 \) and \( g(0,-1) = g(-(0,1)) = -g(0,1) = 0 \).
We show that if \( f \) is differentiable at \((0,0)\), then \( g = 0 \). To this end, let \( T: \mathbb{R}^2 \to \mathbb{R} \)
be the derivative of \( f \) at \((0,0)\), say with matrix \([a \ b] \) in the standard coordinates. Let \( h = (v, w) \neq (0,0) \). Then
\[
\frac{|f(h) - f(0) - Th|}{|h|} = \frac{\sqrt{v^2 + w^2} \cdot g \left( \frac{h}{|h|} \right) - av - bw}{\sqrt{v^2 + w^2}} \\
= \left| g \left( \frac{v}{\sqrt{v^2 + w^2}}, \frac{w}{\sqrt{v^2 + w^2}} \right) - \frac{av + bw}{\sqrt{v^2 + w^2}} \right|.
\]
Approaching \((0,0)\) along the \(x\)-axis,

\[
0 = \lim_{(v,w) \to (0,0)} \left| \frac{g}{\sqrt{v^2 + w^2}} \cdot \frac{w}{\sqrt{v^2 + w^2}} - \frac{av + bw}{\sqrt{v^2 + w^2}} \right|
\]

\[
= \lim_{v \to 0} \left| g \left( \frac{v}{\sqrt{v^2}}, 0 \right) - \frac{av}{\sqrt{v^2}} \right|
\]

\[
= \lim_{v \to 0} \left| g(\text{sgn}(v), 0) - a \cdot \text{sgn}(v) \right|
\]

\[
= |a| \lim_{v \to 0} |\text{sgn}(v)| = |a|
\]

Hence \(a = 0\). Approaching \((0,0)\) along the \(y\)-axis,

\[
0 = \lim_{(v,w) \to (0,0)} \left| \frac{g}{\sqrt{v^2 + w^2}} \cdot \frac{w}{\sqrt{v^2 + w^2}} - \frac{av + bw}{\sqrt{v^2 + w^2}} \right|
\]

\[
= \lim_{w \to 0} \left| g(0, \frac{w}{\sqrt{w^2}}) - \frac{bw}{\sqrt{w^2}} \right|
\]

\[
= \lim_{w \to 0} \left| g(0, \text{sgn}(w)) - b \cdot \text{sgn}(w) \right|
\]

\[
= |b| \lim_{w \to 0} |\text{sgn}(w)| = |b|
\]

Hence \(b = 0\). Now fix \(\theta \in [0, 2\pi]\). Approaching \((0,0)\) along the ray making an angle of measure \(\theta\) with the positive \(x\)-axis,

\[
0 = \lim_{h \to 0} \frac{|f(h) - f(0) - Th|}{|h|} = \lim_{h \to 0} \left| g \left( \frac{h}{|h|} \right) \right| = |g(\cos \theta, \sin \theta)|
\]

Since \(\theta\) was arbitrary, \(g\) is identically \(0\). However, the \(0\) function is differentiable, so \(f\) is not differentiable at \((0,0)\) unless \(g = 0\).

2-5. Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be defined by

\[
f(x,y) = \begin{cases} 
\frac{x|y|}{\sqrt{x^2 + y^2}} & (x,y) \neq 0, \\
0 & (x,y) = 0.
\end{cases}
\]

Show that \(f\) is a function of the kind considered in Problem 2-4, so that \(f\) is not differentiable at \((0,0)\).

**Solution.** Let \(g\) be a real-valued function on the unit circle defined by \(g(a,b) = a|b|\). Clearly \(g\) is continuous, and \(g(0,1) = g(1,0) = 0\) and \(g(-a, -b) = -a - b = -a|b| = -g(a,b)\). If \((x,y) \neq 0\), then

\[
|x,y| \cdot g \left( \frac{x}{|[x,y]|}, \frac{y}{|[x,y]|} \right) = x^2 + y^2 \cdot \frac{x}{\sqrt{x^2 + y^2}} \cdot \frac{|y|}{\sqrt{x^2 + y^2}} = \frac{|x||y|}{\sqrt{x^2 + y^2}}.
\]

Hence \(f\) is a function of the kind considered in Problem 2-4, and since \(g \neq 0\), we conclude that \(f\) is not differentiable at \((0,0)\).

2-6. Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be defined by \(f(x,y) = \sqrt{|xy|}\). Show that \(f\) is not differentiable at \((0,0)\).
Solution. Suppose \( f \) were differentiable, and let \( T : \mathbb{R}^n \to \mathbb{R} \) be the derivative. Let \( [a \ b] \) be the matrix of the linear transformation. We compute

\[
0 = \lim_{h_2 \to 0} \frac{|f(h) - f(0) - Th|}{|h|} = \lim_{h_1 \to 0} \frac{|ah_1|}{|h_1|} = |a|,
\]

\[
0 = \lim_{h_1 \to 0} \frac{|f(h) - f(0) - Th|}{|h|} = \lim_{h_2 \to 0} \frac{|bh_2|}{|h_2|} = |b|.
\]

Hence the derivative, if it exists, is the zero transformation. But

\[
0 = \lim_{h \to 0} \frac{|f(h) - f(0) - Th|}{|h|} = \lim_{h_1 \to 0} \frac{|h_1|}{\sqrt{2}|h_1|} = \frac{1}{\sqrt{2}},
\]

a contradiction.