Related Rates (with trig functions)

Blueprint for how to solve a related rate problem:
(1) Draw a picture! (If you can, or if it hasn’t been made for you.)
(2) What are all relevant, non-constant variables? (i.e., those things which are changing with time)
(3) What rates are given?
(4) What is the desired rate - i.e. the rate we want to know? And under what condition?
(5) What are all possible relations between the variables from step 2?
(6) Implicitly differentiate each relation from step 3 to obtain relations between the rates.
(7) Box or circle the following things:
   - The given rate from step 3.
   - The condition(s) from step 4.
   - Each relation between variables from step 5.
   - Each relation between rates and variables from step 6.
(8) The problem is now just basic (albeit complicated) algebra: solve for the desired rate.

Example (2.8: 16 in the book)

An aircraft spotter observes a plane flying at a constant altitude of 4000 ft toward a point directly above her head. She notes that when the angle of elevation is \( \frac{1}{2} \) radian, it is increasing at a rate of \( \frac{1}{10} \) radian per second. What is the speed of the airplane?
(1) Draw a picture:

![Diagram](image)

where $O$ is the observer and $P$ is the plane.

(2) What are all relevant, non-constant variables? (i.e., those things which are changing with time)

- The elevation angle: $\theta$
- Horizontal position of the plane: $x$

(3) What rates are given?

We know that $\frac{d\theta}{dt} = \frac{1}{10}$ when $\theta = \frac{1}{2}$.

(4) What is the desired rate - i.e. the rate we want to know? And under what condition?

We want to know $\frac{dx}{dt}$ when $\theta = \frac{1}{2}$.

(5) What are all possible relations between the variables from step 2?

The sohcahtoa formula for tangent tells us that

$$\tan \theta = \frac{4000}{x}$$

(6) Implicitly differentiate each relation from step 3 to obtain relations between the rates.

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = -\frac{4000}{x^2} \cdot \frac{dx}{dt}$$

(7) Box or circle the following things:

- The given rate from step 3.
- The condition(s) from step 4.
- Each relation between variables from step 5.
- Each relation between rates and variables from step 6.

(8) The problem is now just basic (albeit complicated) algebra: solve for the desired rate.

First we list all the boxed formulas:

- $\frac{d\theta}{dt} = \frac{1}{10}$
- $\theta = \frac{1}{2}$
\[ \tan \theta = \frac{4000}{x} \]
\[ \sec^2 \frac{d\theta}{dt} = -\frac{4000}{x^2} \cdot \frac{dx}{dt} \]

The third formula tells us that
\[ x = \frac{4000}{\tan \theta} = \frac{4000}{\tan \frac{1}{2}} \]

Now we can plug in our values for \( d\theta/dt, \theta, \) and \( x \) into the fourth formula, and solve for \( dx/dt \).
\[ \sec^2 \left( \frac{1}{2} \right) \cdot \frac{1}{10} = -\frac{4000}{(4000/\tan \frac{1}{2})^2} \cdot \frac{dx}{dt} \]

Solving for \( dx/dt \), we get
\[ \frac{dx}{dt} = -400 \sec^2 \frac{1}{2} \tan^2 \frac{1}{2} = -400 \csc^2 \frac{1}{2} \approx -834.332 \]

**Position/Velocity/Acceleration (with trig functions)**

Recall from last quarter: a position function \( f(t) \) is a function which determines the placement of a moving object \( P \) along a number line. For example, if at time \( t \) the object \( P \) is at the value 3, then \( f(t) = 3 \).

If \( P \) has position function \( f(t) \), what is the object \( P \) doing? (We imagine the number line that the object is the \( y \)-axis, so that the position function gives us the object’s height.)

- \( f(t) = k \) (for some constant \( k \))
  The object is not moving - it is permanently at the value \( k \).
- \( f(t) = t \)
  The object moves up at a constant speed of 1
- \( f(t) = \sin t \)
  The object oscillates between \(-1\) and 1.

The velocity of \( P \) is the derivative \( f'(t) \), while the acceleration of \( P \) is the second derivative \( f''(t) \). The initial position/velocity/acceleration of \( P \) is the value \( f(0)/f'(0)/f''(0) \).

For these examples, we will imagine that the number line is vertical, so that the position function determines height. (On your homework, however, we imagine that the number line that the object is moving on is horizontal.)

**Problem 1**

Consider an object \( P \) with position/height function \( f(t) = \sin(2t) \) for times \( t \in [0, 2\pi] \).
1. What is the initial position of $P$?

$$f(0) = \sin(2 \cdot 0) = 0$$

2. What is the initial velocity of $P$?

Since $f'(t) = 2 \cdot \cos 2t$, we see

$$f'(0) = 2 \cos(2 \cdot 0) = 2$$

3. How is the acceleration of $P$ related to the position $f(t)$ of $P$?

The acceleration of $P$ is $f''(t) = -4 \sin(2t)$. So the acceleration is $-4$ times the position function. I.e.,

$$f'' = -4 \cdot f$$

4. When does $P$ reach a maximum height?

**Problem 2 (Not done in class)**

Consider an object $P$ with position/height function $f(t) = -3 \cos(t)$ for times $t \in [0, 2\pi]$.

1. What is the initial position of $P$?

2. What is the initial velocity of $P$?

3. How is the acceleration of $P$ related to the position $f(t)$ of $P$?

4. When does $P$ reach a minimum height?