Lecture 14
Area and
the Definite Integral

Math 13200

We return to the problem of defining the area of regions of the $x$-$y$ plane.

Since we know how to find the area of polyhedral regions, we use them to define area in general.

**Definition:** Let $R$ be a region in the plane. Suppose that there exists two sequences of polyhedral regions

$$ P_1, P_2, P_3, \ldots $$

$$ Q_1, Q_2, Q_3, \ldots $$

Such that $P_n \subset R$ and $R \subset Q_n$ for each $n$, and

$$ \lim_{n \to \infty} A(P_n) = \lim_{n \to \infty} A(Q_n) $$

Then we say that $R$ is **(Riemann) measurable**, and define its area to be the above common value

$$ A(R) = \lim_{n \to \infty} A(P_n) = \lim_{n \to \infty} A(Q_n) $$

Two facts:

- Not all subsets of the plane are measurable in this sense. For example, the set of all points with rational coordinates is not.

- This definition of area satisfies the five axioms of area stated last lecture. The verification of this fact comes from limit properties, and the corresponding facts for polyhedral regions.

**Example: Circle**

We will find the area of a circle of radius $r$ with this definition. Let $P_n$ be the regular polygon inscribed in the circle, and let $Q_n$ be the regular polygon circumscribed around the circle.
You will show in the homework that the area of $P_n$ is equal to

$$A(P_n) = \frac{1}{2}nr^2 \sin \frac{2\pi}{n}$$

You will also show that

$$\lim_{n \to \infty} A(P_n) = \pi r^2$$

Similarly, we have

$$A(Q_n) = na^2 \tan \frac{\pi}{n}$$

and

$$\lim_{n \to \infty} A(Q_n) = \pi r^2$$

So, we conclude that the area of a circle is given by $\pi r^2$.

**Riemann Sums**

We will use the previous approach to measure areas cut out by the graphs of functions. For the rest of today’s lecture, we will assume our function $f(x)$ is positive.

Specifically, we will look at polyhedron regions that are made up of rectangles:
We will let the length of the rectangles go to 0 to approximate the area. To study such approximations, we will need some definitions.

A partition \( P = \{I_1, \cdots, I_n\} \) of an interval \( I = [a, b] \) is a collection of subintervals
\[
I_1 = [x_0, x_1], \quad I_2 = [x_1, x_2], \quad \cdots \quad I_n = [x_{n-1}, x_n]
\]
with \( a = x_0 \) and \( b = x_n \).

The \( i \)-th subinterval is \( I_i = [x_{i-1}, x_i] \).

For a given partition, the quantity \( \Delta x_i \) is defined to be
\[
\Delta x_i = x_i - x_{i-1} = \text{length of } I_i
\]

The norm \( \|P\| \) of a partition is the maximum of all the \( \Delta x_i \):
\[
\|P\| = \max_i \Delta x_i
\]

Thus, when \( \|P\| \) is small, all the subintervals of the partition are small. When \( \|P\| \) is big, at least one of the subintervals is big.

A sample \( \bar{x}_i \) of a partition \( P = \{I_1, \cdots, I_n\} \) is a choice of point for each interval \( I_i \):
\[
\bar{x}_i \in [x_{i-1}, x_i] = I_i
\]

For any partition, the left sample is the sample with \( \bar{x}_i = x_{i-1} \). The right sample is the sample with \( \bar{x}_i = x_i \). The midpoint sample is the sample with
\[
\bar{x}_i = \frac{x_i - x_{i-1}}{2}
\]

If \( f(x) \) is a function defined on \( I \), and \( P \) is a partition of \( I \), then the min sample is a sample such that
\[
f(\bar{x}_i) = \min_{x \in I_i} f(x)
\]
The **max sample** is defined similarly.

Examples.

First we will just do some examples of partitions/samples without functions (so the min/max samples are irrelevant). Consider the interval $[-4, 10]$. An example of a partition would be

\[ [-4, 2] \quad [2, 2.5] \quad [2.5, 5] \quad [5, 7] \quad [7, 10] \]

What is the norm of this partition? Answer: 6

What is the left sample?

\[ \bar{x}_1 = -4 \quad \bar{x}_2 = 2 \quad \bar{x}_3 = 2.5 \quad \bar{x}_4 = 5 \quad \bar{x}_5 = 7 \]

What is the midpoint sample? What is the right sample?

Consider the function $f(x) = 5 - x^2$ on $I = [-4, 10]$. What is the min sample? The max sample?

Suppose $f(x)$ is defined on an interval $I$, which has partition $P = \{I_1, \cdots , I_n\}$ and a sample $\bar{x}_i \in I_i$ for each $i$. The **Riemann sum** corresponding to $f, P, \bar{x}_i$ is the value

\[ RS_{f,P,\bar{x}_i} = \sum_{i=1}^{n} f(\bar{x}_i) \Delta x_i \]

**Key point 1:**

Some key points:
• The value of the Riemann sum is the area of the polygonal region composed of rectangles with base given by each subinterval $I_i$, and height corresponding to the point $(\bar{x}_i, f(\bar{x}_i))$ on the graph of $f$. Will call this the associated Riemann polygon. Thus, this gives an approximation of the area under the graph of $f$.

• If the sample is the min sample for $f$, then the corresponding Riemann polygon is contained in the area under the graph of $f$. Similarly, the Riemann polygon for the max sample contains the area under the graph of $f$. This mimics what we did with the area of a circle earlier.

• When the norm of the partition is small, the Riemann sum approximates the area under the graph better.

All of this motivates the following definition:

Suppose that $f$ is a function defined on $I = [a, b]$. Suppose we have two sequences of partitions $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ of $I$. For the $P_j$, let $\bar{x}_i$ be the min sample, and for the $Q_j$, let $\bar{x}_i$ be the max sample. If

$$\lim_{j \to \infty} RS_{f, P_j, \bar{x}_i} = \lim_{j \to \infty} RS_{f, Q_j, \bar{x}_i}$$

Then, we say that $f$ is integrable, and we denote the common value be denoted

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\bar{x}_i) \Delta x_i$$

This is called the definite integral from $a$ to $b$. 

• Jordan measure
• Jordan measure of circle
• Jordan measure of area under $y = x^2$.
• Definitions related to Riemann sums: partition, $i$th subinterval, $\Delta x_i$, sample points, sampled partition, norm of a partition
• Riemann sum of a function for a sampled partition, min/max sample (picture)
• Definition of integrable, definite integral; integrals with $a \geq b$.
• Connection between min/max sample and Jordan measure when $f \geq 0$.
• Geometric interpretation when $f$ is non-positive.
• Integrability Theorem
• Negatively oriented integrals
• Interval Additive Property