Introduction and Definition

Almost every mathematical operation has an inverse operation.

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What about differentiation?

**Definition:** We call $F$ an **antiderivative** of $f$ on the interval $I$ if

$$D_x F(x) = f(x)$$

on $I$, that is $F'(x) = f(x)$ for all $x$ in $I$.

We call $F$ an **antiderivative**, not the antiderivative, because there are many antiderivatives. Luckily, they are all closely related.

**Example 1**

Find an antiderivative of $f(x) = \csc^2 x$.

**Solution**

Our knowledge of trig derivatives tells us that

$$D_x (\cot x) = \csc^2 x$$

Thus, $F(x) = - \cot x$ is an antiderivative of $f(x)$. 
Example 2

Find an antiderivative of the function \( f(x) = 4x^3 \) on \( \mathbb{R} \).

Solution

We want a function \( w \)uch that \( F'(x) = 4x^3 \) for all real \( x \). From our experience with differentiation, we know that \( F(x) = x^4 \) is one function.

What about other solutions? \( x^4 + 3 \) also has derivative \( 4x^3 \), since the derivative of a constant is 0.

Unicity of Anti-derivatives

**Theorem** Suppose that \( F_1(x) \) and \( F_2(x) \) are both antiderivatives of \( f(x) \). Then there exists a constant \( C \) such that

\[
F_2(x) = F_1(x) + C
\]

The implication of this theorem is that we only need to find one antiderivative to find them all. All other anti-derivatives are obtained by adding constants.

General Antiderivative

The **general antiderivative** of a function \( f(x) \) is the collection of all antiderivatives of \( f(x) \). It is often written with a “generic constant” \( C \) in the following way: If \( F(x) \) is one anti-derivative of \( f(x) \), then the general antiderivative is

\[
F(x) + C
\]

since all anti-derivatives are equal to \( F(x) + C \) for some constant \( C \).
Notation

We will use two notations for the anti-derivative. We will not use the most common notation, at least until the Fundamental Theorem of Calculus.

The most common way I will write the anti-derivative is as a capital letter for the function. I.e., if \( f, g, h \) are functions, then \( F(x), G(x), H(x) \) will represent anti-derivatives.

I might use the notation \( A_x f(x) \) for the anti-derivative. This mimics the \( D_x f(x) \) notation - where \( D \) and \( A \) stand for derivative and anti-derivative.

Note that just like multiplication and division are inverse, differentiation and anti-differentiation are inverse to each other:

\[
\frac{a}{b} = c \iff a = b \cdot c
\]

\[
A_x f(x) = F(x) \iff f(x) = D_x F(x)
\]

Rules

Power Rule

\[
A_x x^r = \frac{x^r}{r+1} + C
\]

Trig Functions \( \sin \) and \( \cos \)

\[
A_x \sin x = -\cos x + C \quad A_x \cos x = -\sin x + C
\]

Linearity

\[
A_x (k \cdot f(x)) = k \cdot A_x f(x) \quad (1)
\]

\[
A_x (f(x) + g(x)) = A_x (f(x)) + A_x (g(x)) \quad (2)
\]

\[
A_x (f(x) - g(x)) = A_x (f(x)) - A_x (g(x)) \quad (3)
\]
Examples

Find the anti-derivatives of

- $3x^2 + 4x$
- $x^{3/2} - 3x + 14$
- $\frac{1}{t^2} - \sqrt{t}$

Generalized Power Rule

$$A_x[g(x)^r \cdot g'(x)] = \frac{g(x)^{r+1}}{r+1} + C$$

Examples

Find the anti-derivatives of

- $(x^4 + 3x)^{30} \cdot (4x^3 + 3x)$
- $\sin^{10} x \cdot \cos x$