Sketching Graphs

In today’s lecture, we synthesize our pre-calculus knowledge + our knowledge of first derivatives + our knowledge of second derivatives to produce very accurate graphs of functions.

Computer algebra systems and graphing calculators can of course do this much better than we can. But by doing the work ourselves, we get a much clearer picture of how the geometry of the graph of a function reflects its algebraic and infinitesimal properties.

There are many geometric properties of the graph of a function $f(x)$ that we can determine using pre-calculus and calculus. Specifically, when we sketch the graph, we will try to accurately portray:

- the domain and range,
- any symmetry the function has (even/odd or periodicity),
- $x$ and $y$ Intercepts,
- critical points,
- montonocity - increasing and decreasing,
- local minima and maxima,
- concavity,
- inflection points,
- asymptotes, and
- several random points.

Before we go on, we need to recall the pre-calculus notions of graph symmetry: even, odd and periodic.
Symmetry Definitions

Let \( f(x) \) be a function defined on all of \( \mathbb{R} \).

- A function is **even** if \( f(-x) = f(x) \) for all \( x \in \mathbb{R} \).
- A function is **odd** if \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \).
- A function is **periodic** of period \( p \) if \( f(x + p) = f(x) \) for all \( x \).

Examples

- cos and sec are even.
- Constant functions, \( x^2, x^4 + 3, 4x^8 - 2x^5 + x^4 - 1 \) are even.
- sin, cot, csc and tan are odd.
- \( x, x^3, x^5 + 3x, 4x^7 - 2x^5 + x^3 - x \) are odd.
- The trig functions \( \sin x, \cos x, \tan x, \) etc. are all periodic with period \( 2\pi \).
- The function \( x - \lfloor x \rfloor \) is periodic with period 1.

Geometric Interpretation

- An even function’s graph has reflective symmetry in the \( y \)-axis. On the left, we see the graph of
  \[
  f(x) = |x| + x^4 - 3x^2
  \]
  which is an even function.
- An odd function’s graph has rotational symmetry around the origin. In the middle, we see the graph of
  \[
  f(x) = \frac{x^3 + 3x}{x^4 - 3x^2 + 4}
  \]
  which is an odd function.
- A periodic function’s graph has translational symmetry along the \( x \)-axis. On the right, we see the graph of
  \[
  f(x) = \sin(\pi x) - x + \lfloor x \rfloor
  \]
  which is a periodic function with period 2.
Examples

Example 1

Sketch the graph of

\[ f(x) = \frac{3x^5 - 20x^3}{32} \]

Example 2

Sketch the graph of

\[ f(x) = \frac{x^2 - 2x + 4}{x - 2} \]
Example 3

Sketch the graph of

\[ f(x) = \tan x + \cot x \]