Let $\Sigma$ be a finite set. Recall from the previous setup that we have an abelian group $G$ acting on the set $\Sigma^G$ of functions $G \to \Sigma$. For a function $\omega \in \Sigma^G$, this $G$-action is given by
$$(g \cdot \omega)(h) = \omega(g + h), \quad h \in G$$

For today, $G = \mathbb{Z}$, the integers with the group operation of addition. We will think of $\mathbb{Z}$ as parametrizing the action of time. We set $\Omega := \Sigma^\mathbb{Z}$. We would like to put an interesting metric on $\Omega$, to study the dynamics on $\Omega$ of the shift operator $\sigma : \Omega \to \Omega$, to study $\sigma$-invariant measures, and interesting invariant subdynamics of $\sigma$.

We label the elements of $\Sigma$ with integers in order to distinguish them. Thus we write $\Sigma = \{1, \ldots, n\}$. Points of $\Omega$ will be represented as infinite strings of elements of $\Sigma$, together with a decimal point that marks which element of the string corresponds to $0 \in \mathbb{Z}$.

$$\Omega = \{\omega = \ldots \omega_{-2} \omega_{-1} \omega_0 \omega_1 \omega_2 \ldots : \omega_i \in \Sigma\}$$

Our convention will be that the element to the right of the decimal point corresponds to $0 \in \mathbb{Z}$. We also use $\omega_i$ to denote the value of the function $\omega$ at the integer $i$. The $\mathbb{Z}$-action on $\Omega$ has a natural interpretation in this representation. The $\mathbb{Z}$-action is specified by the action of the generator $1$. We will denote the action of $1$ by $\sigma : \Omega \to \Omega$. $\sigma$ satisfies the formula
$$[\sigma(\omega)](n) = \omega(n + 1)$$

In the infinite string notation,
$$\sigma(\ldots \omega_{-2} \omega_{-1} \omega_0 \omega_1 \omega_2 \ldots) = \ldots \omega_{-2} \omega_{-1} \omega_0 \omega_1 \omega_2 \ldots$$

The effect of $\sigma$ is to shift the decimal point one spot to the right, or equivalently, $\omega_0$ is shifted one spot to the left. $\sigma^k$ shifts the decimal point to the right $k$ spots (for $k > 0$) and $\sigma^{-1}$ shifts the decimal point to the left one spot.

A string $\omega$ is periodic if there is some $k \neq 0$ such that $\sigma^k \omega = \omega$. All periodic strings of period $\ell$ are constructed by taking a finite string $j_1 \ldots j_\ell$, $j_i \in \Sigma$, placing the decimal point at the beginning of this string, and then concatenating infinitely many copies of this finite string in both directions. Hence it will appear as
$$\ldots j_1 \ldots j_\ell \cdot j_1 \ldots j_\ell \ldots$$

We want to put a metric on $\Omega$. $\Sigma$ carries a natural metric, the discrete metric, for which the distance between $i, j \in \Sigma$ is given by $\delta_{ij}$, the Kronecker delta that is $1$ if $i = j$, and $0$ otherwise. Up to scaling, this is the only metric on $\Sigma$ which is invariant under all permutations of the set $\{1, \ldots, n\}$. We define a metric $d$ on $\Omega$ by setting, for two strings $\omega, \tau$,
$$d(\omega, \tau) = \sum_{i = -\infty}^{\infty} \frac{1}{2|i|} (1 - \delta_{\omega_i, \tau_i})$$
Exercise: Prove this is a metric on $\Omega = \Sigma^\mathbb{Z}$ which induces the product topology.

The metric $d$ compares the values of $\tau$ and $\omega$ at each integer, giving more weight to values closer to the decimal point. If $\tau_i = \omega_i$ for $|i| \leq m$, then

$$d(\tau, \omega) \leq \frac{1}{2^m}$$

whereas if $\tau_i \neq \omega_i$, then

$$d(\tau, \omega) \geq \frac{1}{2^{|i|}}$$

So two sequences are close if and only if they agree on all sufficiently small integers.

Comment: There is no particular reason for choosing the geometric sequence associated to $1/2$ in the definition of $d$. We could equally well have defined, for any choice of $0 < \lambda < 1$,

$$d_\lambda(\omega, \tau) = \sum_{i=-\infty}^{\infty} \lambda^{|i|}(1 - \delta_{\omega_i \tau_i})$$

What’s important, as we will see later, is that no matter what the choice of $\lambda$ is, the collection of Hölder continuous functions $\Omega \to \mathbb{R}$ remains the same (though possibly with different Hölder exponents depending on $\lambda$).

A basis for the induced topology of this metric is given by cylinder sets. A cylinder set is a clopen subset of $\Omega$ determined by $j \in \mathbb{Z}$, and a finite word $k_0, \ldots, k_\ell$.

$$C(j; k_0, \ldots, k_\ell) = \{ \omega \in \Omega | \omega(j+i) = k_i, i = 0, \ldots, \ell \}$$

The integer $j$ centers us at the $j$th entry of the sequence; $C(j; k_0, \ldots, k_\ell)$ is the set of all sequences which start with the word $k_0 \ldots k_\ell$ from the $j$th spot.

Exercise: Let $\Sigma = \{1, 2\}$ have two elements. Let $\mathcal{C}$ be the standard middle-thirds Cantor set inside of $[0, 1]$. Construct a biHölder homeomorphism between $\Omega$ and $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C} \times \mathcal{C}$ carries the induced metric from the $L^1$ metric on $\mathbb{R}^2$.

Remark: $\Omega$ is always a Cantor set, in the sense that there is a homeomorphism from $\Omega$ to the standard middle-thirds Cantor set. Any metric space $X$ which is compact, totally disconnected, and perfect is homeomorphic to a Cantor set, and $\Omega$ is easily seen to satisfy these properties, no matter what the choice of $\Sigma$ is.

How might one construct an element $\omega$ of the shift space with dense orbit, i.e., such that $\{\sigma^k(\omega) : k \in \mathbb{Z}\} = \Omega$? Two procedures were suggested. First, we could enumerate all finite strings of numbers from $\Sigma$ and concatenate them to form a sequence $\omega$. Second, we could take a countable dense collection of points $\tau_j \in \Omega$, and form $\omega$ by concatenating successively longer strings of numbers from these points $\tau_j$.

We change perspectives now to look at measures on $\Omega$.

$$\mathcal{M}_*(\Omega) = \{ \text{Borel probability measures } \mu \text{ on } \Omega \text{ such that } \sigma_* \mu = \mu \}$$

Recall that $\sigma_* \mu = \mu$ is equivalent to saying that $\mu$ is invariant under $\sigma$,

$$\mu(\sigma^{-1}(A)) = \mu(A) = \mu(\sigma(A))$$

for every Borel set $A$. The second equality requires that $\sigma$ is invertible. Invariance in this context means that the measure of a cylinder set $C(j; k_0, \ldots, k_\ell)$ does not depend on $j$.

The simplest examples of invariant measures for the shift are given by Bernoulli measures. Let

$$\Delta^{n-1} = \left\{ p \in \mathbb{R}^n : p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}$$
be the standard $n-1$ simplex in $\mathbb{R}^n$. A point $p \in \Delta^{n-1}$ will be referred to as a probability vector. Given a probability vector $p$, we define $\mu$ on cylinder sets by

$$
\mu(C(j; k_0, \ldots, k_\ell)) = \prod_{i=0}^{\ell} p_{k_i}
$$

and setting $\mu(\emptyset) = 1$, $\mu(\emptyset) = 0$. We’ve defined $\mu$ on cylinder sets now; we need to extend $\mu$ to be defined on the full Borel $\sigma$-algebra of $\Omega$. The tool we will use for this is the Hahn-Kolmogorov extension theorem.

Some definitions: For a set $X$, a $\sigma$-algebra is a collection $\mathcal{A}$ of subsets of $X$ satisfying

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \rightarrow X \setminus A \in \mathcal{A}$
3. For any countable index set $I$, if $A_i \in \mathcal{A}$ for every $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{A}$.

The collection $\mathcal{A}$ is called an algebra, if it satisfies 1,2, and the weaker condition 3’ that $\bigcup_{i \in I} A_i \in \mathcal{A}$ for any finite index set $I$.

The Hahn-Kolmogorov extension theorem says the following: Let $\mathcal{A}_0$ be an algebra, and let $\mathcal{A}$ be the smallest $\sigma$-algebra containing $\mathcal{A}_0$. Let $\mu_0 : \mathcal{A}_0 \rightarrow [0, 1]$ be a function such that $\mu_0(\emptyset) = 0$, and $\mu_0$ is finitely additive: for any disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$,

$$
\mu_0 \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu_0(A_i)
$$

Suppose further that $\mu_0$ is $\sigma$-additive: whenever we have a countable collection $\{A_i \in \mathcal{A} \}_{i \in I}$ such that $\bigcup_{i \in I} A_i \in \mathcal{A}$, we also have

$$
\mu_0 \left( \bigcup_{i \in I} A_i \right) = \sum_{i \in I} \mu_0(A_i)
$$

Then $\mu_0$ extends to a measure on $\mathcal{A}$. Further, this extension is unique if $\mu_0$ is $\sigma$-finite.

We can check that the measure $\mu$ constructed on the algebra generated by the cylinder sets satisfies these properties and so extends uniquely to a measure on the Borel $\sigma$-algebra of $\Omega$. The uniqueness property of the Hahn-Kolmogorov extension theorem also means that, to check that $\mu$ is $\sigma$-invariant, it suffices to check that $\mu$ is $\sigma$-invariant when restricted to unions of cylinder sets, which is clear.

The second example is Markov measures. These are defined by two objects. As before, we have a probability vector $p \in \Delta^{n-1}$, but we also have an $n \times n$ stochastic matrix $P$. $P$ being stochastic means that $P_{ij} \in [0, 1]$ for every $i, j$ and $\sum_{j=1}^{n} P_{ij} = 1$. We assume also that $p \cdot P = p$, so that $p$ is a left eigenvector of $P$ with eigenvalue 1. Written out in sums, this says that

$$
\sum_{i=1}^{n} p_i P_{ij} = p_j
$$

for every $i, j$. $P$ has the property that for every $q \in \Delta^{n-1}$, $p \cdot P \in \Delta^{n-1}$.

If we take a stochastic matrix $P$ and assume that there is some $N > 0$ such that all of the entries of $P^N$ are strictly positive, then it follows from the Perron-Frobenius theorem that there is a unique left eigenvector $p \in \Delta^{n-1}$ whose eigenvalue is the leading eigenvalue of $P$. We can deduce that the leading eigenvalue of $P$ is 1 by observing that the column vector of all 1’s is a right eigenvector of $P$, and then noting that eigenvalues are preserved under transposition.
Using the pair \((P,p)\) of a stochastic matrix and an associated probability vector, we can define a measure \(\mu\) on cylinder sets by

\[
\mu(C(j;k_0,\ldots,k_\ell)) = p_{k_0} \cdot \prod_{i=0}^{\ell-1} P_{k_i,k_{i+1}}
\]

**Exercise**: Prove that \(\mu\) gives rise to a shift-invariant measure on \(\Omega\).