Differential Topology, Winter 2016, Final

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This final exam was posted online on Friday, March 11, and is due by 11:30 a.m. Wednesday, March 16. Collaboration is not allowed, nor is the use of outside materials and textbooks. Milnor-Stasheff, Warner and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. This problem is about vector bundles on spheres. Throughout the problem, let $\xi$ be an oriented $\mathbb{R}^n$ bundle over $S^m$.

1. Let $D^m_\pm$ denote the northern and southern hemisphere of $S^m$. Show that $\xi$ can be trivialized over each of $D^m_\pm$, and the difference of trivialization gives a map $c : S^{m-1} \to \text{GL}^+(n, \mathbb{R})$ (this map is called a clutching function).
2. Show that any two trivializations of $\xi$ over $D^m_\pm$ give rise to homotopic clutching functions. Conversely, show that any clutching function arises for some bundle $\xi$. Thus deduce that there is a bijection between $\pi_{m-1}(\text{GL}^+(n, \mathbb{R}))$ and the set of isomorphism classes of oriented $\mathbb{R}^n$-bundles over $S^m$.
3. Conclude that there is a natural isomorphism $\pi_{m-1}(\text{GL}^+(n, \mathbb{R})) = \pi_m(\mathbb{R}_+^2)$ where $\mathbb{R}_+^2$ is the Grassmannian of oriented $\mathbb{R}^n$s in $\mathbb{R}^\infty$.

(Remark: in fact, $\text{GL}^+(n, \mathbb{R})$ is homotopic to the loop space $\Omega G^+_n$. Also note that since $G^+_n$ is a connected double cover of the unoriented Grassmannian $G_n$, they have the same homotopy groups in dimension $> 1$. Thus in fact $\pi_{m-1}(\text{GL}(n, \mathbb{R})) = \pi_m(G_n)$ too.)

Problem 2. (1) If $M$ is a closed, orientable manifold of dimension $n$ where $n$ is odd, show that the Euler characteristic $\chi(M)$ is equal to zero.
2. Using obstruction theory (for instance), show that a closed, orientable manifold of odd dimension admits a nowhere zero vector field.
3. Give an example of an orientable $\mathbb{R}^3$ bundle over $S^4$ which has zero Euler class but does not admit a nowhere zero section.

Problem 3. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth immersion. Give $\mathbb{R}^2$ the usual $x, y$ coordinates, and consider the form $xdy$.

1. The algebraic area enclosed by $\gamma$ is defined as follows. If $\Gamma : D^2 \to \mathbb{R}^2$ is a smooth map which restricts to $\gamma$ on the boundary (i.e. such that $\Gamma|\partial D = \gamma$) then we can pull back the area form $dx \wedge dy$ on $\mathbb{R}^2$ under $\Gamma$ and integrate it over $D^2$. Then
   \[
   \text{algebraic area enclosed by } \gamma = \int_{D^2} \Gamma^* dx \wedge dy
   \]
   Show that this does not depend on the choice of $\Gamma$, and is equal to $\int_\gamma xdy$. Furthermore, if $\gamma$ is an embedding, so that $\gamma(S^1)$ encloses a smooth disk, show that this agrees with the “usual” area enclosed by $\gamma$ (up to sign).
2. Let $\alpha := dz - xdy$ be a 1-form on $\mathbb{R}^3$ with coordinates $x, y, z$. We let $\xi$ be the 2 dimensional subbundle of $T\mathbb{R}^3$ consisting of vectors in the kernel of $\alpha$ (pointwise). Show (e.g. by using Frobenius’ theorem) that $\xi$ is nowhere integrable.
3. With notation as above, let $\gamma : S^1 \to \mathbb{R}^2$ be a smooth immersion. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be obtained by forgetting the $z$ coordinate. A lift of $\gamma$ is a smooth map $\hat{\gamma} : S^1 \to \mathbb{R}^3$ such that $\pi \circ \hat{\gamma} = \gamma$. We would like to find a lift $\hat{\gamma}$ of $\gamma$ that satisfies further the condition that $\alpha(\hat{\gamma}') = 0$; i.e. $\hat{\gamma}$ is everywhere tangent to $\xi$. Show that a lift of $\gamma$ satisfying this further condition exists if and only if
if the algebraic area enclosed by $\gamma$ is zero. Draw some examples of smoothly immersed circles in the plane which enclose algebraic area zero.

(4) Show that for any two points $p, q \in \mathbb{R}^3$ there is a smooth curve $\beta : [0, 1] \to \mathbb{R}^3$ with $\alpha(\beta') = 0$ pointwise, and such that $\beta(0) = p$ and $\beta(1) = q$.

Problem 4. Let $S \subset \mathbb{R}^3$ be a smoothly embedded surface. Then we can define a connection on the tangent bundle $TS$ as follows.

(1) The tangent space $T\mathbb{R}^3$ is naturally a trivial $\mathbb{R}^3$ bundle, by using the vector space structure on $\mathbb{R}^3$. Show that the restriction $T\mathbb{R}^3|_S$ is in a natural way a trivialized $\mathbb{R}^3$ bundle over $S$, and show there is a natural connection $\tilde{\nabla}$ on this bundle in which the “constant” sections are parallel.

(2) Let $\pi : T\mathbb{R}^3|_S \to TS$ denote orthogonal projection fiberwise (i.e. for each point $s \in S$ this is the map that orthogonally projects $T_s\mathbb{R}^3$ to $T_sS$). By abuse of notation, we let $\pi$ also denote the induced map from sections of $T\mathbb{R}^3|_S$ to sections of $TS$. Show that $\nabla := \pi \circ \tilde{\nabla}$ defines a connection on $TS$.

(3) For $S$ the round unit 2-sphere in $\mathbb{R}^3$ compute the curvature of this connection as a matrix of 2-forms (expressed in local coordinates).

Problem 5. Let $M$ be a closed oriented $n$-manifold, and let $\xi$ be an oriented $\mathbb{R}^n$-bundle over $M$, so that the Euler class $e(\xi)$ is a well-defined element of $H^n(M; \mathbb{Z})$. Suppose that $\xi$ admits a flat connection. Show that if $\tau$ is a (smooth) triangulation of $M$, then $|e(\xi)([M])|$ is no greater than the number of $n$-dimensional simplices of $M$.

(hint: show that we can find a section of $\xi$ which looks “linear” on each simplex with respect to a local trivialization by parallel sections. Now observe that a linear map from $\Delta^n$ to $\mathbb{R}^n$ in general position has at most one zero.)

Bonus problem: Give an example to show that $e(\xi)$ does not need to be zero.