1 Recalling the definition of a limit

First, let’s recall the definition of a limit—both the informal and the formal versions.

We say that \( \lim_{x \to \infty} f(x) = \ell \) if

- **Informal:** For every version of “close to”, we can choose some meaning for “large” such that if \( x \) is “large,” then \( f(x) \) is “close to” \( \ell \).

- **Formal:** For all real \( \varepsilon > 0 \), there exists \( N \) such that for all \( x > N \),
  \[
  |f(x) - \ell| < \varepsilon.
  \]

The following table shows the correspondence between the informal version and the formal version.

<table>
<thead>
<tr>
<th>Informal</th>
<th>Formal</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>For every version of “close to”</td>
<td>For every ( \varepsilon &gt; 0 )</td>
<td>Each ( \varepsilon ) gives us a meaning for “close to”—namely, “within ( \varepsilon ).”</td>
</tr>
<tr>
<td>we can choose some meaning for “large”</td>
<td>there exists ( N )</td>
<td>When we’ve chosen ( N ), we say that “large” means “bigger than ( N ).”</td>
</tr>
<tr>
<td>such that if ( x ) is “large,”</td>
<td>such that if ( x &gt; N )</td>
<td>As we’ve said, ( x ) is “large” if ( x &gt; N ).</td>
</tr>
<tr>
<td>then ( f(x) ) is “close to” ( \ell ).</td>
<td>then (</td>
<td>f(x) - \ell</td>
</tr>
</tbody>
</table>
2 Newton’s “definition of a limit”

Consider the following statement from Isaac Newton’s seminal work, the Philosophiae Naturalis Principia Mathematica:

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

This was centuries before mathematicians came up with the correct definition of a limit in order to build the “skyscraper” of analysis. Newton was trying to build his “cloud castle” of Calculus. It’s kind of hard to see in the middle of a cloud, so it’s no wonder he was confused: he thought he was proving a theorem rather than stating a definition.

Nevertheless, this statement has some of the key aspects of the definition of a limit. Newton understood that it is not enough just to say that one quantity “approaches” another. He put in a key phrase: approaches nearer than by any given difference. In other words, when we say that “f(t) approaches ℓ,” we really mean that f(t) becomes arbitrarily close to ℓ. In more modern language, Newton’s “difference” would probably be called ε. We would say that for any given ε, f(t) must approach to within ε of ℓ.

And he incorporated another key understanding—how exactly does this “becoming close” depend on t? Newton saw t as time. What we called f(t), he might have called “the value of f once the time t has passed.” Letting t get larger is, for him, simply letting a lot of time pass. And when we think about it this way, we come to the following realization. In order for f(t) to approach ℓ “nearer than a given difference,” f(t) must become nearer than that difference in finite time. In other words, there is a time N, after which f(t) becomes—and remains—within ε of ℓ.

Thus, in Newton’s language, we have the following definition of limit:

We say that a function f(t) (where t represents time) has a limit ℓ if for any given difference ε, within finite time, the quantity f(t) approaches—and remains—nearer to ℓ than by ε.

Exercise. Relate this definition to the formal definition of a limit by making a table like the one at the end of Section 7.

3 Computing limits when they exist

One of the interesting things about limits (as well as other major characters we will meet in the study of Calculus) is that the usual methods of computing them look practically nothing like the definition. The following “theorem” (it’s really a bunch of theorems stated at the same time) is essentially copied from page 68 of the textbook, and is quite useful for evaluating limits. It gives situations in which limits behave exactly as you might hope.
**Theorem.** ("Main Limit Theorem") In the following equations, if the right side makes sense, then the left side also makes sense and is equal to the right side.

1. \[ \lim_{x \to \infty} k = k \]
2. \[ \lim_{x \to \infty} \frac{1}{x} = 0 \]
3. \[ \lim_{x \to \infty} [f(x) + g(x)] = \left[ \lim_{x \to \infty} f(x) \right] + \left[ \lim_{x \to \infty} g(x) \right] \]
4. \[ \lim_{x \to \infty} [f(x) - g(x)] = \left[ \lim_{x \to \infty} f(x) \right] - \left[ \lim_{x \to \infty} g(x) \right] \]
5. \[ \lim_{x \to \infty} [f(x) \cdot g(x)] = \left[ \lim_{x \to \infty} f(x) \right] \cdot \left[ \lim_{x \to \infty} g(x) \right] \]
6. \[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)} \]
7. \[ \lim_{x \to \infty} [f(x)]^n = \left[ \lim_{x \to \infty} f(x) \right]^n \]

"The right side makes sense" means, for now, that the limits in question exist (as real numbers) and there is no division by 0.

This theorem can be proved from the definition of the limit. The proofs are not even that difficult. But the only way they can ever be interesting is when you do them yourself. Watching someone else do them is terribly boring, so I’ll skip the proofs—at least for now—and move straight to discussing how to use the theorem to actually compute limits.

**Warning.** If you use this theorem (typically, repeated applications of this theorem) to compute a limit, then you will have shown, in the process, that the limit exists. However, if you try to apply this theorem, and end up with something that makes no sense, you will not have shown that the original limit does not exist.

**Example.** (Example 2, p. 78 in the textbook) Compute

\[ \lim_{x \to \infty} \frac{x}{1 + x^2} . \]

In particular, show that it exists.

**Solution.** The most obvious thing to try here is to apply Rule 6 which would tell us that

\[ \lim_{x \to \infty} \frac{x}{1 + x^2} = \frac{\lim_{x \to \infty} x}{\lim_{x \to \infty} 1 + x^2} . \]
assuming that the righthand side makes sense. Unfortunately, the right hand side does not make sense: the limits on the righthand side do not exist.\footnote{In a more sophisticated point of view that we will adopt later, the numerator and the denominator are both $\infty$. But $\infty/\infty$ still does not make sense, as we will discuss.}

A more successful way to solve this problem is to first divide both the top and the bottom by the highest power of $x$ that appears in the denominator.

\[
\lim_{x \to \infty} \frac{x}{1 + x^2} = \lim_{x \to \infty} \frac{x}{1 + x^2} \cdot \frac{1/x^2}{1/x^2} \quad \text{(algebra)} \quad (1)
\]

\[
= \lim_{x \to \infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2} + 1} \quad \text{(algebra)} \quad (2)
\]

\[
= \frac{\lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} \left(\frac{1}{x^2} + 1\right)} \quad \text{(Rule 6)} \quad (3)
\]

\[
= \frac{\lim_{x \to \infty} \frac{1}{x}}{(\lim_{x \to \infty} \frac{1}{x})^2 + \lim_{x \to \infty} 1} \quad \text{(Rules 3, 7)} \quad (4)
\]

\[
= \frac{0}{0^2 + 1} \quad \text{(Rules 2, 1)} \quad (5)
\]

\[
= 0. \quad (6)
\]

To the right of each line is written the justification: why do we know it is equal to the previous line (assuming it is defined)?

A few words should be said on how we actually know the limits exist. If we actually want to be careful here, our knowledge of the limits goes from the bottom of the stack of formulas to the top. Because line $\text{(5)}$ makes sense, the theorem tells us that line $\text{(4)}$ makes sense and is equal to it. Because line $\text{(4)}$ makes sense, the theorem tells us that line $\text{(3)}$ makes sense and is equal to it. And so on, all the way up to the top (which is what we cared about to begin with).

**General procedure for computing limits of rational functions:**

A rational function, as you may recall, is a function of the form

\[f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0}.\]

When faced with a function like this and asked to compute $\lim_{x \to \infty} f(x)$, here is a procedure that often works:

1. Multiply the numerator and denominator both by $1/x^k$.
2. Use the rules of the “Main Limit Theorem” to “distribute” the limit signs. Bring them further and further “inside” the formula, until all the limits are of the form $\lim_{x \to \infty} 1/x = 0$ or $\lim_{x \to \infty} k = k$.\footnote{In a more sophisticated point of view that we will adopt later, the numerator and the denominator are both $\infty$. But $\infty/\infty$ still does not make sense, as we will discuss.}
Assignment 6 (due Friday, 14 October)

Section 1.5, problems 1-9. Remember: It is not enough to get the right answer. You have to convince the reader that your answer is right. Problems 2, 4, 6, and 8 will be graded carefully.

Assignment 7 (due Monday, 17 October)

Do the exercise at the end of Lecture 8, Section 2 on Newton’s “definition of a limit.”

In the textbook, Section 1.5, Problems 15, 16, and 18. Problems 16 and 18 will be graded carefully.

Complete the attached worksheet on graphing piecewise-defined functions.