1 Logistics: Review session on Friday

Since reading period starts tomorrow, class on Friday will not introduce any new material. Instead, I will devote the class to answering students’ questions. I expect most of the time to be spent on going over how to solve different kinds of problems, but I will also take questions about what sorts of things are and are not fair game for the exam.

Attendance is not required, but I think the class will be more helpful for everyone if a lot of people show up. I want people to do well on the final, and it is very frustrating for me when people miss something that they could have gotten right if they had only asked me to explain it.

Chaofan and Jay will not be holding tutorials tomorrow. Seth, however, will (in Pick 022); everyone is welcome to attend, whether or not you are in Seth’s tutorial normally.

2 Maxima and minima—motivation applications

We will be studying how to use calculus (specifically, derivatives) to find points at which a function is maximized or minimized. This sort of thing has many practical applications. For instance, we can ask

- What price should we sell lemonade at in order to make the most profit? (Profit is a function of price; we want to select price to maximize it.)

- What shape should a rectangle be to fence in the largest possible area with a fixed amount of fence? (The area of the rectangle is a function of its length; we want to maximize it.)

- What path should a pipeline follow under a river to minimize the cost of building it? (The cost is a function of the path; we want to minimize this function.)

We won’t get to solve these sorts of problems in this lecture (and thus not until next quarter), but I will show you the mathematical tools that are used to solve them.
3 Maxima and minima—the theory

We’re going to spend a few minutes talking about the basic theory (theorems and such) before seeing the applications.

**Definition.** Let $f$ be a function. The maximum value of $f$ is a value $M$ such that

(i) $f$ attains the value $M$; i.e., there is some $x_0$ such that $M = f(x_0)$; and

(ii) $M \geq f(x)$ for all $x$ in the domain of $f$.

The minimum value of $f$ is a value $m$ such that

(i) $f$ attains the value $m$; i.e., there is some $x_0$ such that $m = f(x_0)$; and

(ii) $m \leq f(x)$ for all $x$ in the domain of $f$.

An extreme value of $f$ is a value $y$ that is either the maximum or the minimum value of $f$.

**Warning.** Maximum and minimum values need not exist; consider the following two cases.
This function has no maximum because it attains arbitrarily large values.

In both of the cases above, the “issue” was that there were points at which the function had no finite limit. Specifically,

$$\lim_{x \to 0} f(x) = \infty, \quad \text{while} \quad \lim_{x \to 1} g(x) \text{ does not exist.}$$

This yields plausibility to the following theorem:

**Theorem.** Let \( f \) be a continuous function on a closed interval \([a, b]\). Then \( f \) has a minimum and a maximum.

We won’t even try to prove this. For the function \( f \) above, the function was defined on \((0, 1]\), but 0 was missing from the domain—the interval was not closed. For \( g \), the function was not continuous.

The points where minimum and maximum values might take place are called **critical points**. More precisely,

**Definition.** Let \( f \) be a function defined on an interval \([a, b]\) and \( x_0 \) a point in its domain. We say that \( x_0 \) is a critical point of \( f \) if any of the following holds:
• $x_0$ is an endpoint of the interval (i.e., $x_0 = a$ or $x_0 = b$); or
• $f'(x_0)$ does not exist; or
• $f'(x_0) = 0$.

The last type of critical point, where $f'(x_0) = 0$, is in some sense the most interesting sort of critical point to find (find the derivative $f'$, then solve for $f'(x) = 0$). But the other two kinds should not be forgotten, since they are absolutely necessary to make the following theorem true.

**Theorem.** Let $f$ be a continuous function with domain a closed interval $[a, b]$. Then the only points where $f$ could possibly equal its extreme values are the critical points.

**Idea of proof.** We prove the contrapositive. Suppose $x_0$ is not a critical point. We will show that $f(x_0)$ is not an extremal value of $f$.

Since $x_0$ is not a critical point, $x_0$ is differentiable and $f'(x_0) \neq 0$. In other words, $f$ has a tangent line at $x_0$ that is not horizontal. Thus, for $x$ sufficiently close to $x_0$, $f(x)$ is contained in a narrow cone about the tangent line.

Since the tangent line is not horizontal, if we make the cone sufficiently narrow, we can ensure that the values of $f$ immediately to the right of $x_0$ (if the slope is positive) or immediately to the left of $x_0$ (if the slope is negative) are above $f(x_0)$. Since $x_0$ is not a critical point, it is not an endpoint of the domain, so $f$ does have values immediately to the left and right of $x_0$. Hence, $f(x_0)$ is not an maximum of $f$.

Similar reasoning shows that $f(x_0)$ is not a minimum value of $f$.
4 Maxima and minima: example

In other words, if we know \( f \) is a continuous function on \([a, b]\), then the following procedure will allow us to find the minima and maxima of \( f \) on \([a, b]\):

1. Find the critical points of \( f \) (all three kinds).
2. Evaluate \( f \) at each of the critical points.
3. The largest of the resulting values is the maximum value of \( f \) on \([a, b]\).
   The least of the resulting values is the minimum value of \( f \) on \([a, b]\).

**Example 1.** Find the critical points, minimum, and maximum for the function \( f \) given by
\[
f(x) = \frac{1}{3}x^3 - x
\]
on the closed interval \([-2.5, 1.5]\).