1 Implicit Differentiation: How to differentiate a function we don’t know

All of the “exercises” for differentiation so far have been based on differentiating formulas. However, many of the rules for differentiation (most especially, the chain rule) are much more general than this: they deal with differentiating functions. And, as you may recall, kind of the whole point of functions is that they are not necessarily given by formulas. We have not really explored this very far, because most of the functions we could talk about were, in fact, given by formulas. But there is another way: we can define a function as a solution to something. For instance, the √ function is really defined by

\[ \sqrt{x} = \text{the nonnegative number } y \text{ such that } y^2 = x. \]

In other words, \( \sqrt{x} \) is just a fancy way of writing “the (nonnegative) solution to the equation \( y^2 = x \).” And we can go back to this basic definition to differentiate the square root function:

**Example 1.** Suppose \( y = \sqrt{x} \). Find an expression for \( \frac{dy}{dx} \).

**Solution.** We assume, first of all, that \( \sqrt{x} \) is in fact differentiable; without this assumption, there is not much we can do. We then go back to the basic equation that \( y^2 = x \) and apply the Chain Rule:

\[
\begin{align*}
y^2 &= x \\
\frac{d}{dx} y^2 &= \frac{d}{dx} x \\
2y \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{2y} = \frac{1}{2\sqrt{x}}.
\end{align*}
\]

Thus, assuming that the function \( f \) taking \( x \mapsto \sqrt{x} \) is differentiable, its derivative \( f' \) is necessarily given by

\[ f'(x) = \frac{1}{2\sqrt{x}}. \]
Exercise 2. Use implicit differentiation to show that if \( y = -\sqrt{x} \), then
\[
\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}.
\]

Solution.

Note that there is a problem we never dealt with here: we never actually showed that \( f(x) = \sqrt{x} \) is differentiable. We only figured out what its derivative must be, if the derivative exists. There are a couple ways to solve this problem.

- It is possible to compute the derivative of \( \sqrt{x} \) directly from the definition of the derivative (i.e., as the limit of the difference quotient). This is done earlier in the textbook. However, this is a way to avoid using implicit differentiation; what we really want is a way to show that implicit differentiation works.

- There is a theorem called the “Implicit Function Theorem” that states, roughly, that if implicit differentiation gives a reasonable answer, then the equation in question does in fact have a solution \( y = f(x) \) where \( f \) is a differentiable function. This is kind of like the Main Limit Theorem: If the process gives a reasonable answer, then we know that must be the right answer; but if the process does not give a reasonable answer, we don’t know anything.

The Implicit Function Theorem may seem to be the answer to our problems, but there are subtleties even here. First, the actual statement of the theorem is something that I find confusing, so I very much doubt that you want to see it. Second, while the Implicit Function Theorem can guarantee that some solutions are differentiable (in this case, \( f(x) = \sqrt{x} \) and \( f(x) = -\sqrt{x} \) are both solutions to \( f(x)^2 = x \) that are differentiable for \( x > 0 \)), there will also be other solutions that are not differentiable. For instance, if \( f \) is the function defined by
\[
f(x) = \begin{cases} 
\sqrt{x} & \text{if } 0 < x \leq 1, \\
-\sqrt{x} & \text{if } x > 1,
\end{cases}
\]
then $y = f(x)$ is also a solution to the equation $y^2 = x$ for all $x > 0$, but $f$ is not even continuous, much less differentiable. We will not try to explain why the Implicit Function Theorem applies for some “solutions,” but not to others. Instead, we will adopt a “third way”:

- Ignore the difficulties and just assume implicit differentiation works. Any function we encounter “naturally” in this course\(^1\) is going to work out just fine.

In essence, we’ve reached a point where the skyscraper just gets too convoluted to deal with, so we’re going to continue walking on clouds.

There’s one more very important result we want to obtain using implicit differentiation. Recall that we proved the Power Rule, $D_x(x^n) = nx^{n-1}$, whenever $n$ is an integer. We’re now going to that this holds, not just for integers, but for rational numbers.

**Theorem.** (Power Rule for rational exponents) Let $r$ be any rational number. Then

$$D_x(x^r) = rx^{r-1}.$$ 

**Incomplete Proof.** Since $r$ is a rational number (i.e., a “ratio” of two integers), we may write

$$r = \frac{p}{q},$$

for some integers $p, q$, where $q \neq 0$. By definition, $y = x^{p/q}$ is a solution to the equation

$$y^q = x^p.$$

\(^1\)That is, any function that has not been explicitly designed to cause problems.
Applying implicit differentiation, together with the power rule for integer exponents, we see that

\[
qy^{q-1} \frac{dy}{dx} = px^{p-1}
\]

[\text{Here is a detailed derivation process omitted for brevity.}]

\[
\frac{dy}{dx} = \frac{px^{b-1}}{qy^{q-1}}
= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^r)^{q-1}}
= r \cdot x^{(p-1) - r(q-1)}
= r \cdot x^{p-1 - (p/q)(q-1)}
= r \cdot x^{p-1 - p + p/q}
= r \cdot x^{1 - (p/q)}
= r \cdot x^{r-1}.
\]

The key point of this proof is that we could apply the power rule to \(x^p\) and \(y^q\), because we already knew the power rule for integer exponents, and \(p, q\) are integers. This proof is incomplete in that we have not really turned implicit differentiation into a rigorous technique, so we can’t use it in “real” proofs.

I commented at one point that calculus is “supposed” to work exactly the same for rational and irrational numbers. Thus, it seems peculiar that we have a rule that only seems to work for rational numbers. In fact, as it turns out, the Power Rule does hold for all real exponents—rational or irrational. There’s even a nice, elegant proof that does not care whether \(r\) is rational or irrational. Unfortunately, this proof uses logarithms, so we won’t see it for some time (if at all). Thus, for now, all our powers will be rational.

2 Some potential pitfalls: numbers, functions, and expressions

When I first introduced functions, I made a big deal of the fact that \(f\) is a function, but \(f(x)\) is just a number (albeit one that we do not yet know). In terms of this distinction, differentiation is something we do to functions, not numbers. Thus, \(Df\), the “derivative of \(f\),” is a function, but \(Df(x)\) would be the “derivative of a number,” which does not make any sense. Unfortunately, this distinction has become somewhat blurred when we write things like

\[
\frac{d}{dx}(x^2 + 1).
\]

What we really mean here is “the derivative of the function that maps \(x \mapsto x^2 + 1\).” The \(x\) in \(d/dx\) tells us that \(x\) is just a “dummy variable,” and so the input is really just a function. When we write the answer as \(2x\), it is even harder
to tell that we mean “the function mapping $x \mapsto 2x$” rather than simply “the number $2x$.”

So far, this section has been entirely theoretical, but there is a practical, computational issue as well. Suppose someone asks you to calculate the derivative of $x^2 + 1$ at $x = 2$. You may be tempted to substitute in $x = 2$ before differentiating, which would be a disaster. You’d be differentiating a number rather than a function; you’d probably try to treat it as the constant function $2^2 + 1 = 5$, and end up getting derivative 0 since the derivative of any constant function is zero.

To be honest, I hope that none of you would make this particular error, because this example is fairly straightforward. But when you deal with more complicated relations—say, $u$ and $v$ are both functions of $t$, $y$ is a function of $u$, and you have some equation that involves all four letters $t, u, v, y$—it can be easy to lose track of whether you are dealing with functions or numbers “underneath.” A good rule of thumb here is the following:

**Rule of Thumb.** First, do all your differentiating. Then, and only then, start treating variables as numbers.

For instance, if you are asked to find the derivative of $x^2 + 1$ at $x = 2$, you should first differentiate (obtaining $2x$) and then substitute in $x = 2$ (obtaining 4, the correct answer). Like any rule of thumb, this one has occasional exceptions. The only truly reliable way to stay out of trouble is to know what you are doing: to know, at each step of your argument, whether $x^2 + 1$ really means “the number $x^2 + 1$” or “the function that maps $x \mapsto x^2 + 1$.” However, trying to keep track of this can be quite confusing, and I think the Rule of Thumb above will probably serve you well.
Assignment 21 (due Wednesday, 23 November)

[Note: If you won’t be in class on the day before Thanksgiving, then some time before class, put the homework in my mailbox (in the Eckhart basement). Also, send me an email so that I know you have done this.]

Section 2.6, Problems 11–12 and 20–21. Problems 12 and 21 will be graded carefully.

Section 2.7, Problems 3–6 and 19–20. The even-numbered problems will be graded carefully.

("Semi-bonus problem") Suppose that

\[ \lim_{x \to c^-} f(x) = \ell = \lim_{x \to c^+} f(x); \]

i.e., both the one-sided limits are defined, and they are equal. Use the \( \varepsilon-\delta \) definition of the limit to show that the two-sided limit is also defined and equal to \( \ell \), i.e., that

\[ \lim_{x \to c} f(x) = \ell. \]

The technique involved should be similar to that used to give \( \varepsilon-\delta \) proofs for piecewise linear functions.

This is a “semi-bonus problem” in the following sense:

- If you do not seriously attempt it, you will not receive full credit on your homework.
- If you seriously attempt it, you will receive full credit for it (although your actual homework grade will, of course, depend on the other homework problems).
- If you get it right, you will receive a bonus point on the homework.

Assignment 22 (due Monday, 28 November)

Section 2.5, Problems 19 and 20.

Section 2.6, Problems 23 and 38. Both of these will be graded carefully.

Section 2.7, Problems 8, 21, and 37. Problems 21 and 37 will be graded carefully.

Section 2.8, Problem 1.