1 Analysis is about inequalities, not equations

Traditional mathematics is about equations—determining when two quantities are equal. To “calculate” a quantity means, typically, to find an equal quantity that is easier to work with. For instance, when we convert a fraction to a decimal, we obtain the same number in a form that is easier to add to other numbers.

However, this notion breaks down when we try to deal with irrational numbers like $\sqrt{2}$. No matter how many digits of $\sqrt{2}$ we calculate, we will never find a decimal number equal to $\sqrt{2}$. The most we can do is to approximate $\sqrt{2}$. Thus, for instance, when we state that the first few digits of $\sqrt{2}$ are 1.414, we are really stating that

$$1.414 \leq \sqrt{2} \leq 1.415;$$

since all quantities are positive, we can square them to obtain the equivalent inequality

$$1.414^2 \leq 2 \leq 1.415^2;$$

a statement that can be tested without already “knowing” the value of $\sqrt{2}$.

When we want to deal with real numbers (and in particular, with irrational numbers), we almost always end up dealing with inequalities and approximations rather than actual equations. Thus, we are going to spend some time reviewing how exactly inequalities may be manipulated.

A word on things to come: the “skyscraper” of analysis is all about inequalities. However, once we get to the “cloud castle” of calculus, we will be back to caring mostly about equations. Thus, somehow, in the process of climbing to the top of the skyscraper, the inequalities get translated back into equalities. This is done using rules like the following:

**Theorem.** (to be proved later in the course) Let $x$ be a real number. If we want to show that $x = 0$, it suffices to show the following: for every positive number $\varepsilon$,

$$|x| < \varepsilon.$$
Typically, when you see the symbol $\varepsilon$ (Greek letter epsilon), you should think “small positive number.” This is purely psychological: the statement would be just as correct if you replaced every $\varepsilon$ with a $y$. Nevertheless, this “psychological” choice of variable can provide an important guide for intuition. When you see a statement like the theorem above, you should get the following idea:

“If we can do a good enough job of showing that $x$ is really close to zero, we’ll know that $x$ is actually equal to zero.”

### 2 Rules for manipulating inequalities

If you read Section 0.2 of the textbook, you’ll see a lot of talk about “solving” inequalities. The homework problems will use this term, so you’ll need to make sure you understand what the authors mean by it. However, I prefer to think of “manipulating” inequalities rather than “solving” them. For instance, if you use the authors’ methods to “solve” the inequality

$$x^2 < 2,$$

you’ll get something like

$$-\sqrt{2} < x < \sqrt{2}.$$  

However, since $\sqrt{2}$ is hard to calculate, the initial inequality may be easier to work with than the “solved” version.

Nevertheless, the basic tools are the same whether you want to “solve” inequalities or simply “manipulate” them. Unfortunately, these tools, i.e., rules for manipulation, tend to be more about nitpicking than interesting ideas. I’ve distributed a handout of rules that you should use for reference. Here are a few “traps” you may be tempted to run into, if you’re used to solving equations rather than inequalities:

- “I can multiply both sides by the same number.” **ISSUE:** You need to check the sign first. If you’re multiplying by a positive number, you’re fine. But if you’re multiplying by a negative number, you need to reverse the direction of the inequality sign.

- “I can square both sides.” **ISSUE:** This only works if both sides are positive.

- “If I have an inequality like $(x-a)(x-b) < 0$, where a product is compared to zero, I can split it into the factors: $x-a < 0$, $x-b < 0$.” **ISSUE:** What you can actually say in this particular case is that $x-a$ and $x-b$ have opposite signs. In other words, one is positive and the other is negative. Quadratic inequalities are more complicated than quadratic equations.
If there’s an “interesting idea” in manipulating inequalities, it’s this: in some situations (for instance, in quadratic inequalities), we divide into cases, connected by words like AND and OR. At this point, we are not only doing algebraic manipulations. We are also playing around with the logical relationships among the different inequalities. Here’s an example:

**Example.** (Example 3, Section 0.2 in text) Consider the inequality $x^2 - x < 6$. Much as in the case of quadratic equations, we start out by making one side zero and factoring the other side:

$$x^2 - x < 6$$
$$x^2 - x - 6 < 0 \quad \text{(subtract 6 from both sides)}$$
$$(x - 3)(x + 2) < 0 \quad \text{(factor)}$$

Now, this single inequality is equivalent to the statement that $x - 3$ and $x + 2$ have opposite signs. In other words,

$$((x - 3 < 0) \text{ AND } (x + 2 > 0)) \text{ OR } ((x - 3 > 0) \text{ AND } (x + 2 < 0))$$

We analyze these two cases separately.

**Case 1:**

$$x - 3 < 0 \quad \text{AND} \quad x + 2 > 0$$
$$x < 3 \quad \quad x > -2.$$ 

A shorthand for $(x < 3 \text{ AND } x > -2)$ is

$$-2 < x < 3.$$ 

**Case 2:**

$$x - 3 > 0 \quad \text{AND} \quad x + 2 < 0$$
$$x > 3 \quad \quad x < -2.$$ 

There are no values of $x$ such that $x > 3 \text{ AND } x < -2$. A shorthand for a statement that is never true is $0 = 1$.

**Combining the two cases:** We see that our initial inequality is equivalent to the statement

$$-2 < x < 3 \quad \text{OR} \quad 0 = 1.$$ 

In other words, it is true precisely when at least one of the following is true:

(i) $-2 < x < 3$

(ii) $0 = 1$

Since $0 = 1$ is *never* true, it follows that $x^2 - x < 6$ precisely when $-2 < x < 3$. 

3
3 Assignment due Friday, September 30

Read “A Bit of Logic” and “Quantifiers” on pp. 4–6.
Problem Set 0.1, numbers 45, 46, 63, and 64. Problems 45 and 46 will be graded carefully.

Read pp. 8–9.
Problem Set 0.2, numbers 3, 4, and 12. Problems 4 and 12 will be graded carefully. DO NOT use the quadratic formula on problem 12.

**Bonus Exercise.** Show that the following three conditions on $x$ are equivalent:

(i) $x < \sqrt{2}$.
(ii) $x^2 < 2$ or $x < 0$.
(iii) There exists $y$ such that $(x < y$ AND $y^2 < 2)$.