The process of taking a derivative, often called “differentiation,” is extremely important. Moreover, unlike many important things in mathematics, differentiation is actually possible to do. Any time you have a function given by a formula, the rules in this lecture will allow you to find its derivative.

These rules need to be memorized. Ideally, they should become so ingrained that you can use them without having to think about them.

1 The “easy rules”

There are a few “easy” rules for differentiation.

**Theorem.** (Constant rule) If $f$ is the function defined by $f(x) = c$, where $c$ is a (constant) real number, then $f'(x) = 0$ for all $x$.

**Proof.** This is a special case of the $mx + b$ rule we proved last time, but let’s do it again anyway.

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= 0.
\end{align*}
\]

**Theorem.** (Sum rule) If $f$ and $g$ are differentiable functions, then

\[(f + g)' = f' + g'.\]

In words, “the derivative of a sum is the sum of the derivatives.”
Proof. We apply one of the limit definitions of the derivative:

\[
(f + g)'(x) = \lim_{h \to 0} \frac{(f + g)(x + h) - (f + g)(x)}{h}
= \lim_{h \to 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h}
= \lim_{h \to 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h}
= f'(x) + g'(x)
= (f' + g')(x).
\]

Since this holds for all \(x\) at which \(f\) and \(g\) are defined, we have the equality of functions

\[(f + g)' = f' + g'.\]

\[\square\]

Theorem. (Multiplication by a constant) If \(f\) is a differentiable function of \(x\) and \(c\) is a (constant) real number, then

\[
\frac{d}{dx}(cf(x)) = c \frac{df}{dx}.
\]

Proof. This is, again, a special case of the \(mx + b\) thing. This time, we’re going to derive it from the product rule.

\[
\frac{d}{dx}(cf(x)) = c \frac{df}{dx} + f(x) \frac{d}{dx}(c)
= c \frac{df}{dx} + f(x) \cdot 0 \quad \text{(constant rule)}
= c \frac{df}{dx}.
\]

\[\square\]

Theorem. (Difference rule) If \(f\) and \(g\) are differentiable functions, then \((f - g)' = f' - g'\).

Proof.

\[
(f - g)' = (f + (-1) \cdot g)'
= f' + ((-1)g)'
= f' + (-1)g' \quad \text{(multiplication by a constant)}
= f' - g'.
\]

\[\square\]

2 The power rule; polynomials

The power rule is fairly easy, but a bit less intuitive than the “easy rules.” It was mentioned briefly at the end of the last lecture.
Theorem. When \( n \) is a positive integer,

\[
\frac{d}{dx} x^n = nx^{n-1}.
\]

(Actually, this theorem applies whenever \( n \) is a real number, but we won’t be able to prove that for some time.)

To understand the proof of the Power Rule, we need a technique called \textit{mathematical induction}. Suppose we have a condition \( P(n) \) on \( n \). The “induction principle” says that to show \( P(n) \) is true whenever \( n \) is a positive integer, we can do show the following:

1. \( P(1) \) is true.
2. Whenever \( P(n) \) is true, then \( P(n+1) \) is also true.

Thus, \( P(1) \) is true; since \( P(1) \) is true, \( P(2) \) is also true; since \( P(2) \) is true, \( P(3) \) is also true; and so on.

One standard metaphor here is that in step 2, we set up a chain of dominoes; in step 1, we knock over the first one, which then knocks over the second one, which then knocks over the second one, etc.

Proof. Let \( P(n) \) be the statement that \( D_x(x^n) = nx^{n-1} \); this is a condition on \( n \).

1. We first show that \( P(1) \) is true, i.e., that \( D_x(x) = 1 \):

\[
\frac{d}{dx}(x) = \lim_{h \to 0} \frac{(x+h) - h}{h} = \lim_{h \to 0} \frac{h}{h} = 1,
\]

as desired.

2. We now show, using the product rule, that whenever \( P(n) \) is true, then \( P(n+1) \) is also true.

\[
\frac{d}{dx} x^{n+1} = \frac{d}{dx}(x \cdot x^n) = x \frac{d}{dx}(x^n) + x^n \frac{d}{dx}(x) = x \cdot nx^{n-1} + x^n \cdot 1 = nx^n + x^n = (n+1)x^n.
\]
Thus, by induction, $P(n)$ is true for every positive integer $n$. In other words, for every positive integer $n$,

$$
\frac{d}{dx} x^n = nx^{n-1}. \quad \square
$$

Using the power rule, together with the “easy rules,” we can, in principle, compute the derivative of any polynomial.

**Example 1.** Differentiate $x^2 - 4x + 1$.

**Solution.**

$$
\frac{d}{dx} (x^2 - 4x + 1) = \frac{d}{dx} (x^2) - \frac{d}{dx} (4x) + \frac{d}{dx} (1) \quad \text{(sum rule)}
\begin{align*}
&= \frac{d}{dx} (x^2) - 4 \frac{d}{dx} (x) + 0 \quad \text{(constant multiple; constant)}
&= 2x - 4 \cdot 1 + 0 \quad \text{(power rule)}
&= 2x - 4.
\end{align*}
$$

**Example 2.** Differentiate $2x^3 - \frac{1}{2}x^2 - x + \frac{17246}{937}$.

**Solution.**

$$
\frac{d}{dx} \left(2x^3 - \frac{1}{2}x^2 - x + \frac{17246}{937}\right) = 2 \cdot 3x^2 - \frac{1}{2} \cdot 2x - 1 + 0 \\
= 6x^2 - x - 1. \quad \square
$$

### 3 Proof of the product rule

Recall the product rule,

$$
\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx},
$$

and the (non-rigorous) infinitesimal derivation:

$$
d(uv) = (u + du)(v + dv) - uv \\
= uv + u dv + v du + du dv - uv \\
= u dv + v du \\
\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}.
$$
We are now going to show how to prove the product rule rigorously. Pay attention to how what we are doing rigorously corresponds to the non-rigorous infinitesimal method.

**Theorem.** Suppose that \( u \) is a function of \( x \) such that \( du/dx|_{x=x_0} \) exists. Likewise, suppose that \( v \) is a function of \( x \) such that \( dv/dx|_{x=x_0} \) exists. Then the derivative of the product \( uv \) at \( x_0 \) exists, and

\[
\frac{d}{dx}(uv) \bigg|_{x=x_0} = u_0 \frac{dv}{dx} \bigg|_{x=x_0} + v_0 \frac{du}{dx} \bigg|_{x=x_0}.
\]

**Proof.** First, we write \( \Delta(uv) \) in terms of \( \Delta u \) and \( \Delta v \):

\[
\Delta(uv) = uv - u_0v_0
= (u_0 + \Delta u)(v_0 + \Delta v) - u_0v_0
= u_0v_0 + u_0\Delta v + v_0\Delta u + \Delta u\Delta v - u_0v_0
= u_0\Delta v + v_0\Delta u + \Delta u\Delta v.
\]

[Notice how closely this resembles the infinitesimal version.] Now, we apply the definition\(^1\) of the derivative as a limit:

\[
\frac{d}{dx}(uv) \bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{u_0\Delta v + v_0\Delta u + \Delta u\Delta v}{\Delta x}
= \lim_{\Delta x \to 0} \frac{u_0}{\Delta x} \frac{\Delta v}{\Delta x} + v_0 \frac{\Delta u}{\Delta x} \frac{\Delta v}{\Delta x} \frac{\Delta u}{\Delta x}
= u_0 \frac{dv}{dx} \bigg|_{x=x_0} + v_0 \frac{du}{dx} \bigg|_{x=x_0} + \left( \frac{du}{dx} \bigg|_{x=x_0} \right) \left( \frac{dv}{dx} \bigg|_{x=x_0} \right) \cdot 0
= u_0 \frac{dv}{dx} \bigg|_{x=x_0} + v_0 \frac{du}{dx} \bigg|_{x=x_0}.
\]

Note the trick on the third line that was used to show that \( \Delta u\Delta v/\Delta x \to 0 \):

\[
\frac{\Delta u\Delta v}{\Delta x} = \frac{\Delta u\Delta v}{(\Delta x)^2} \cdot \Delta x = \frac{\Delta u}{\Delta x} \cdot \frac{\Delta v}{\Delta x} \cdot \Delta x \to \frac{du}{dx} \cdot \frac{dv}{dx} \cdot 0 = 0
\]

as \( \Delta x \to 0 \). This (sort of) gives a justification for the infinitesimal idea that \( du\,dv = 0 \).

4 The Chain Rule

Arguably the most important of all of these rules is the chain rule, which tells us how to take derivatives of compositions of functions. It states that if \( f \) and \( g \) are differentiable functions, then

\[
(f \circ g)'(x) = f'(g(x)) \cdot g'(x).
\]

\(^1\)Or rather, one of the equivalent definitions.
We can give a non-rigorous, infinitesimal derivation as follows: One (non-rigorous) definition of the derivative is that, if \( y = f(x) \), then \( f'(x) \) is the number such that
\[
dy = f'(x)dx.
\]
Now, suppose that \( y = f(u) \) and \( u = g(x) \), so that \( y = f(u) = f(g(x)) \). Then we have \( dy = f'(u)du \) and \( du = g'(x)dx \), so
\[
dy = f'(u)du
\]
\[
= f'(u)g'(x)dx
\]
\[
= f'(g(x))g'(x)dx.
\]
Hence,
\[
\frac{dy}{dx} = f'(g(x))g'(x).
\]

**Example 3.** Differentiate \((x + 1)^{500}\).

**Solution.**
\[
\frac{d}{dx} (x + 1)^{500} = 500(x + 1)^{499} \cdot \frac{d}{dx} (x + 1)
\]
\[
= 500(x + 1)^{499} \cdot 1
\]
\[
= 500(x + 1)^{499}.
\]

It would have been possible, but very hard, to differentiate this by expanding out all 501 terms of the polynomial and then applying the techniques of the first section.
Assignment 16 (due Wednesday, 9 November)

Section 2.2, Problems 11–14 and 55–58. Follow the instructions. Remember, these problems (except for 57 and 58) are more about how you find the derivative than what derivative you find. The even-numbered problems will be graded carefully (although a certain amount of leeway will be provided on problem 58).

Section 2.3, Problems 11–14 and 23–26. (Hint: 23–26 are easier if you use the product rule.) The even-numbered problems will be graded carefully.

Translate the statement
\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0) \]
into \(\varepsilon\)-\(\delta\) language. (Hint: when you see \(f'(x_0)\), treat it like \(\ell\). Also, treat \(\Delta y/\Delta x\) as a function of \(\Delta x\).) Then, use the resulting statement to prove the following:

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } |\Delta x| < \delta, \text{ then } \Delta y \text{ is within } \varepsilon|\Delta x| \text{ of } f'(x_0)\Delta x. \]

You will need to handle \(\Delta x = 0\) as a separate case. This statement is a rigorous version of the statement that “When \(\Delta x\) is small, then \(\Delta y\) is approximately \(\frac{dy}{dx}\Delta x\).”

Assignment 17 (due Friday, 11 November)

From Section 2.3:

• Problems 5–8. Do each problem two ways—using the limit definition of your choice, and using the rules of differentiation (including the Chain Rule, if you find it helpful).

• Problems 17–20.

• Problems 31–32. Do not FOIL out the products; instead, use the product rule for differentiation.

The even-numbered problems will be graded carefully.

Section 2.5, Problems 1–4. Make sure it is clear, from your answer, how you are using the Chain Rule (see, for instance, Example 3 at the end of Lecture 18). Problems 2 and 4 will be graded carefully.

Give an \(\varepsilon\)-\(\delta\) proof for each of the following:

1. Let \(f\) be the function defined by

\[ f(x) = \begin{cases} 
7x - 3 & \text{if } x \leq 0, \\
-\frac{1}{2}x - 3 & \text{if } x > 0.
\end{cases} \]

Show that \(\lim_{x \to 0} f(x) = -3\).
2. Let \( f \) be the function defined by
\[
f(x) = \begin{cases} \frac{1}{7}x - \frac{18}{7} & \text{if } x < 3, \\ \frac{1}{6}x - \frac{7}{2} & \text{if } x \geq 3. \end{cases}
\]
Show that \( \lim_{x \to 3} f(x) = -3 \).

3. Let \( f \) be the function defined by
\[
f(x) = \begin{cases} -\frac{1}{2}x + \frac{3}{2} & \text{if } x < -3, \\ 4 & \text{if } x = -3, \\ 3x + 12 & \text{if } x > -3. \end{cases}
\]
Show that \( \lim_{x \to -3} f(x) = 3 \).

Problems 1 and 3 will be graded carefully.

Suppose \( y = f(x) \) and \( f(x_0) = y_0 \). A purely \( \varepsilon-\delta \) version of the statement that “\( f \) is continuous at \( x_0 \)” is given as follows:

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } |x - x_0| < \delta, \text{ then } |y - y_0| < \varepsilon.
\]

Use this definition to prove the following fact:

Suppose that
\[
u = f(x),
\]
\[
u_0 = f(x_0),
\]
\[
y = g(u) = g(f(x)), \quad \text{and}
\]
\[
y_0 = g(u_0) = g(f(x_0)).
\]

If \( f \) is continuous at \( x_0 \) and \( g \) is continuous at \( u_0 \), then \( g \circ f \) is continuous at \( x_0 \).

This will be graded carefully.

**Part of Assignment 18**

Assignment 18 will include giving \( \varepsilon-\delta \) proofs of the following:

1. Let \( f \) be the function defined by
\[
f(x) = \begin{cases} -8x + 10 & \text{if } x \leq 1, \\ 3x - 1 & \text{if } x > 1. \end{cases}
\]
Show that \( \lim_{x \to 1} f(x) = 2. \)
2. Let $f$ be the function defined by

$$f(x) = \begin{cases} 
-4x + 5 & \text{if } x < 1, \\
-\frac{1}{2}x + \frac{3}{2} & \text{if } x > 1.
\end{cases}$$

Show that $\lim_{x \to 1} f(x) = 1$.

3. Let $f$ be the function defined by

$$f(x) = \begin{cases} 
-8x - 2 & \text{if } x < 0, \\
-2 & \text{if } x = 0, \\
\frac{1}{7}x - 2 & \text{if } x > 0.
\end{cases}$$

Show that $\lim_{x \to 0} f(x) = -2$.

**Test II around Wednesday, 16 November**

Experienced teachers of Math 131 tell me that you will probably have a lot of papers and the like due around the time of the test, and consequently will not have a lot of time to study for it. Thus, I suggest you start studying now. You may also want to think in terms of “practicing” rather than “studying”: redoing old quiz and homework problems (without looking at the solutions, if you have them, until afterwards) may be more helpful than simply reading over them.