1 Some example computations

We’re going to do compute some derivatives as functions using the definition

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

**Example.** Suppose that \( f(x) = x \). Compute a formula for the function \( f' \).

**Solution.**

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{(x + h) - x}{h} \\
    &= \lim_{h \to 0} \frac{h}{h} \\
    &= \lim_{h \to 0} 1 \\
    &= 1.
\end{align*}
\]

**Example.** Suppose that \( f(x) = mx + b \). Since \( y = f(x) \) is a line, the tangent line will be the line itself; its slope, of course, is \( m \). Thus, we may suppose that \( f'(x) = m \) for all \( x \). Prove this using the limit definition.
Solution.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{m(x + h) + b - (mx + b)}{h}
= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h}
= \lim_{h \to 0} \frac{mh}{h}
= \lim_{h \to 0} m
= m.
\]

Example. Let \( f \) be the function defined by \( f(x) = x^2 + x - 3 \). Compute a formula for the function \( f' \).

Solution.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{(x + h)^2 + (x + h) - 3 - x^2 - x + 3}{h}
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - 3 - x^2 - x + 3}{h}
= \lim_{h \to 0} \frac{2xh + h^2 + h}{h}
= \lim_{h \to 0} \frac{(2x + h + 1)h}{h}
= \lim_{h \to 0} 2x + h + 1
= 2x + 1.
\]

Example. Let \( f \) be the function defined by \( f(x) = \frac{1}{x} \). Compute a formula for the derivative of \( f \) (except at \( x = 0 \), of course).
Solution.

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x + h)}{x(x + h)} \]

\[ = \lim_{h \to 0} \frac{x - (x + h)}{hx(x + h)} \]

\[ = \lim_{h \to 0} \frac{-h}{hx(x + h)} \]

\[ = \lim_{h \to 0} \frac{-1}{hx(x + h)} \]

\[ = -\frac{1}{x^2}. \]

\[ \square \]

Notation. It can be rather tiresome to write, for instance, “the derivative of the function \( f \) defined by \( f(x) = x^2 + x - 3 \).” In the future, we will sometimes abbreviate this by

\[ \frac{d}{dx} (x^2 + x - 3). \]

2 Product rule

Suppose that we have \( u \) and \( v \), two functions of \( x \). Suppose we know how to calculate the derivatives \( \frac{du}{dx} \) and \( \frac{dv}{dx} \). We can use this to calculate the derivative of the product \( u \cdot v \), by means of the product rule.

Warning. It may be tempting to write that

\[ \frac{d}{dx} (u \cdot v) = \frac{du}{dx} \cdot \frac{dv}{dx}. \]

This is not true. For instance, suppose \( u(x) = 2 \) and \( v(x) = x \). Then

\[ \frac{d}{dx} (u \cdot v) = \frac{d}{dx} (2 \cdot x) = 2, \]

since \( y = 2x \) is a line of slope 2. However, the “naive product rule” would give us

\[ \frac{d}{dx} (2 \cdot x) = \frac{d}{dx} (2) \cdot \frac{d}{dx} (x) = 0 \cdot 1 = 0. \]

The naive product rule gives the wrong answer.

Leibniz gave a cute derivation of the product rule using infinitesimals. The first equation in this proof may seem a bit confusing at first; I’ll explain it
afterwards, but if I give it now, the proof will not seem so “cute.” Remember, the key “fact” about infinitesimals is that if you multiply two of them together, you get something “doubly infinitesimal,” which we typically consider equal to zero. In particular, \( du \, dv = 0 \).

\[
\begin{align*}
  d(uv) &= (u + du)(v + dv) - uv \\
  &= uv + udv + vdu + du dv - uv \\
  &= udv + vdu.
\end{align*}
\]

Dividing through by \( dx \), we see that

\[
\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.
\]

Now, the promised explanation of the first line: we have two functions \( u \) and \( v \) of \( x \). But we really have three functions: the one we care about is the function \( f \) defined by \( f = uv \), i.e.,

\[
f(x) = u(x) \cdot v(x).
\]

Thus,

\[
\begin{align*}
  df &= f(x + dx) - f(x) \\
  &= u(x + dx)v(x + dx) - u(x)v(x).
\end{align*}
\]
Recall that
\[ du = u(x + dx) - u(x), \quad \text{hence} \]
\[ u + du = u(x + dx). \]

Similarly, \( v + dv = v(x + dx) \), and so we have
\[ df = u(x + dx)v(x + dx) - u(x)v(x) \]
\[ = (u + du)(v + dv) - uv. \]

(By an abuse of notation, we’re writing things like \( u \) for \( u(x) \) when it suits us to do so.)

**Example.** Use the product rule to find (in this order) the derivatives of \( x^2 \), \( x^3 \), and \( x^4 \) with respect to \( x \).

**Solution.**
\[
\frac{d}{dx} x^2 = \frac{d}{dx} (x \cdot x) \\
= x \frac{dx}{dx} + x \frac{dx}{dx} \\
= x + x \\
= 2x.
\]
\[
\frac{d}{dx} x^3 = \frac{d}{dx} (x \cdot x^2) \\
= x \frac{d}{dx} (x^2) + x^2 \frac{d}{dx} (x) \\
= x \cdot 2x + x^2 \cdot 1 \\
= 2x^2 + x^2 \\
= 3x^2.
\]
We just calculated \( \frac{d}{dx} (x^2) = 2x \), so this is equal to
\[
= x \cdot 2x + x^2 \cdot 1 \\
= 2x^2 + x^2 \\
= 3x^2.
\]
\[
\frac{d}{dx} x^4 = \frac{d}{dx} (x \cdot x^3) \\
= x \cdot \frac{d}{dx} (x^3) + x^3 \frac{d}{dx} (x) \\
= x \cdot 3x^2 + x^3 \cdot 1 \\
= 3x^3 + x^3 \\
= 4x^3.
\]
\( \square \)
You may start to notice a pattern here. This pattern will continue: if we calculate on out to \( \frac{d}{dx}x^{n-1} \), we’ll find that it is equal to \((n-1)x^{n-2}\). Using this fact, we find that

\[
\frac{d}{dx}x^n = \frac{d}{dx}(x \cdot x^{n-1}) \\
= x \cdot \frac{d}{dx}(x^{n-1}) + x^{n-1} \frac{d}{dx}(x) \\
= x \cdot (n-1)x^{n-2} + x^{n-1} \cdot 1 \\
= (n-1)x^{n-1} + x^{n-1} \\
= nx^{n-1},
\]

so the pattern always keeps going. (This is a version of “proof by induction.”)
Assignment 15 (due Monday, 7 November)

Section 2.2, Problems 5–8 and 51–54. Follow the instructions. Remember, these problems are more about how you find the derivative than what derivative you find. The even-numbered problems will be graded carefully.

Section 2.3, Problems 1-4. These will not be graded carefully.

Draw a picture that explains why the difference quotient
\[
\frac{f(x + h) - f(x)}{h}
\]
gives the slope of a secant line to the curve \( y = f(x) \). Hint: your intuition may like this problem better if you think in terms of \( x_0 \) rather than \( x \). This problem will be graded carefully.

Assignment 16 (due Wednesday, 9 November)

Section 2.2, Problems 11–14 and 55–58. Follow the instructions. Remember, these problems (except for 57 and 58) are more about how you find the derivative than what derivative you find. The even-numbered problems will be graded carefully (although a certain amount of leeway will be provided on problem 58).

Section 2.3, Problems 11–14 and 23–26. (Hint: 23–26 are easier if you use the product rule.) The even-numbered problems will be graded carefully.

Translate the statement
\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)
\]
into \( \varepsilon-\delta \) language. (Hint: when you see \( f'(x_0) \), treat it like \( \ell \). Also, treat \( \Delta y/\Delta x \) as a function of \( \Delta x \).) Then, use the resulting statement to prove the following:

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } |\Delta x| < \delta, \text{ then } \Delta y \text{ is within } \varepsilon \text{ of } f'(x_0)\Delta x.
\]

You will need to handle \( \Delta x = 0 \) as a separate case. This statement is a rigorous version of the statement that “When \( \Delta x \) is small, then \( \Delta y \) is approximately \( \frac{dy}{dx} \Delta x \).”

Test II around Wednesday, 16 November

Experienced teachers of Math 131 tell me that you will probably have a lot of papers and the like due around the time of the test, and consequently will not have a lot of time to study for it. Thus, I suggest you start studying now. You may also want to think in terms of “practicing” rather than “studying”: redoing old quiz and homework problems (without looking at the solutions, if you have them, until afterwards) may be more helpful than simply reading over them.