1 Some alternate ways to state the limit definition of the derivative

For this section, we’re going to use the notation \( f'(x_0) \) rather than \( \frac{dy}{dx}_{x=x_0} \). Our basic definition of the derivative has been

\[
f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]

(1)

One key to mastering mathematics is being able to move facilely among different ways of saying the same thing; which way you want to say it may depend on what you want to use it for. We’re going to review some other ways to write the definition of the derivative, using the various relations among \( x, x_0, \Delta x, y, \Delta y, f(x_0), \ldots \).

First, observe that

\[
\Delta y = y - y_0 = f(x) - f(x_0) \quad \text{and} \quad \Delta x = x - x_0.
\]

Thus,

\[
\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0},
\]

and

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \ell
\]

\[\iff \forall \varepsilon \gt 0, \exists \delta \gt 0 \text{ s.t. if } 0 \lt |\Delta x - 0| \lt \delta, \text{ then } \left| \frac{\Delta y}{\Delta x} - \ell \right| \lt \varepsilon \]

\[\iff \forall \varepsilon \gt 0, \exists \delta \gt 0 \text{ s.t. if } 0 \lt |x - x_0| \lt \delta, \text{ then } \left| \frac{f(x) - f(x_0)}{x - x_0} - \ell \right| \lt \varepsilon \]

\[\iff \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \ell.\]
In other words, an alternate definition for the derivative is given by
\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \] (2)

This definition highlights the feature that the derivative only depends on what is happening to \( f \) near \( x_0 \). If we look at a different function \( g \) that cannot be distinguished from \( f \) near \( x_0 \), then \( f \) and \( g \) will have the same derivative at \( x_0 \); i.e., \( f'(x_0) = g'(x_0) \).

Another way to state the definition of the derivative is to express \( \Delta y \) in terms of \( x_0 \) and \( \Delta x \), rather than \( x_0 \) and \( x \).
\[ \Delta y = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0), \]

since \( x = x_0 + \Delta x \). Thus, we have
\[ f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \]

Making the traditional change of notation \( \Delta x = h \), we find that
\[ f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \] (3)

The expression inside the limit is the infamous “difference quotient.”

## 2 Derivative as a function

In the definition of (3), one feature is that there are no appearances of the letter \( x \) except in the variable \( x_0 \). Thus, we can rename \( x_0 \) as \( x \), obtaining
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

The interesting feature here is that when we rewrite the definition this way, it becomes obvious that we have defined more than a number \( f'(x_0) \); we have defined a function \( f' \).

There’s a subtlety here that confused me when I first saw this sort of thing. It involves the interplay of intuition and rigorous mathematics. Intuitively, when we write \( x \), we think of it as a variable—something that is allowed to range over many different numbers. On the other hand, when we write \( x_0 \), we think of this as a particular value of \( x \), a particular number; we just don’t happen to know what number it is. These intuitions are valuable. However, it is equally valuable to realize that these intuitions have absolutely no reflection in the rigorous mathematics. As far as the pure logic is concerned, \( x \) and \( x_0 \) are both variables, and that’s all there is to it. So whenever we have a statement
that involves only one, we can substitute the other, and get an equally true expression that feels very different, intuitively.

This is typical of a certain kind of reasoning that appears sometimes in mathematics. First, you let your intuition guide you, as we did (more or less) in defining the derivative. Then you do something with rigorous mathematics to change the statement into something equivalent, but that feels intuitively very different. At this point, you may feel like your head wants to explode: your intuition is screaming that what you’ve done can’t possibly be right, but you can’t see any flaws in your logic. It may be tempting to give up and think about something else. But instead, you may force yourself to stay on task, to turn the thing over and over in your head until you either find a flaw in the logic, or find a way of thinking about it that your intuition will accept. Depending on the difficulty of the thing in question, resolving the conflict may take moments, hours, days, weeks, months, or years. But the longer you spend puzzling over it, the greater will be your feeling of enlightenment when it finally “clicks.”

On the other hand, some of you may be thinking that it was obvious that the derivative is a function. You may even feel a bit smug about the fact that this “revelation” was clear to you from the beginning. Perhaps you should. But I think it is more likely that you were not following my lectures closely, but were instead thinking about the derivative in terms you have learned in the past. Or perhaps you never really understood the intuition of \( x_0 \) as a “fixed value we don’t know,” versus \( x \) as a “variable.” Either way, I suggest you review the previous buildup to the definition of the derivative. Try to understand with your whole mind—both logic and intuition. If you succeed, you may get a part of the revelatory moment that you will otherwise have been cheated of.

Now, enough philosophizing. Since we’ve established that the derivative \( f' \) is a function, there are two obvious sorts of questions:

1. How do we find a formula for the function, if one exists?

2. How do we characterize the function, even if it does not have a formula we can write down?

We’ll spend a lot of time on both of these, but in light of the homework I’ve assigned you for Friday, I’m going to spend the rest of this lecture on a version of the second problem. Specifically: If someone gives you a graph of the function, how do you graph its derivative? We’ll approach this mainly through examples. My plan (which I may or may not have time for) is to give you a few minutes to try the following examples on your own, and then we will go over them together.
Example. The graph of a function $f$ is given on the left. On the right, sketch the graph of the function $f'$. Remember: above each point $x$ on the $x$-axis, the value of $f'$ should be the slope of the tangent line to $f$ at $x$. If $f$ does not have a unique tangent line at $x$, then $f'(x)$ will not exist.
3 Local nature of the limit (and derivative)

I’m probably not going to have time to really go over this section in the lecture, but I would feel like I would not be fulfilling my responsibilities as a Math 131 teacher if I did not at least mention it in the lecture notes.

Recall that, in the most vague terms, the statement

\[ \lim_{x \to x_0} f(x) = \ell \]

means something like “when \( x \) is near \( x_0 \), then \( f(x) \) is near \( \ell \).” Thus, it seems like this limit should only depend on “what \( f \) is doing near \( x_0 \).” In particular, it should only depend on how \( f \) behaves on an interval \((x_0 - \Delta x, x_0 + \Delta x)\).

The way we say that the limit “only depends on what \( f \) is doing near \( x_0 \)” is that if we replace \( f \) by a different function \( g \) that “looks the same near \( x_0 \),” then we are guaranteed to get the same answer. More precisely, we have the following theorem:

**Theorem.** Suppose that \( f \) and \( g \) are two functions. Let \( \Delta x \) be positive. If \( f \) and \( g \) are defined and agree on the interval \((x_0 - \Delta x, x_0 + \Delta x)\), then

\[ \lim_{x \to x_0} f(x) \text{ exists if and only if } \lim_{x \to x_0} g(x) \text{ exists.} \]

Moreover, if the two limits exist, then they are equal.

**Proof.** Assume that

\[ \lim_{x \to x_0} f(x) = \ell. \]

We will then show that \( \lim_{x \to x_0} g(x) = \ell \).

Let \( \varepsilon > 0 \) be given.
Since \( \lim_{x \to x_0} f(x) = \ell \), there exists \( \delta_1 > 0 \) such that if \( 0 < |x - x_0| < \delta_1 \), then \( |f(x) - \ell| < \varepsilon \). Set \( \delta = \min\{\delta_1, \Delta x\} \).

Assume \( 0 < |x - x_0| < \delta \). Since \( |x - x_0| < \delta \leq \Delta x \), we know \( f(x) = g(x) \). Consequently,

\[
|g(x) - \ell| = |f(x) - \ell| < \varepsilon,
\]

since \( 0 < |x - x_0| < \delta \leq \delta_1 \).

Therefore,

\[
\lim_{x \to x_0} g(x) = \ell,
\]

as claimed.

Similar reasoning shows that, if

\[
\lim_{x \to x_0} g(x) = \ell,
\]

then \( \lim_{x \to x_0} f(x) = \ell \).

Since one definition for the derivative is

\[
\frac{dy}{dx}_{x = x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]

the theorem tells us that the derivative of \( f \) at \( x_0 \) depends only on how \( f \) behaves near \( x_0 \).
Assignment 14 (due Friday, 4 November)

Section 2.2, Problems 37–44. The even-numbered problems will be graded carefully.

For each of Problems 1–4 in Section 2.2, do the following steps:

(a) Find the indicated derivative using infinitesimals.

(b) Find the indicated derivative using the limit definition.

(c) Graph the function together with the tangent line at the indicated point.

Problems 2 and 4 will be graded carefully.

Assignment 15 (due Monday, 7 November)

Section 2.2, Problems 5–8 and 51–54. Follow the instructions. Remember, these problems are more about how you find the derivative than what derivative you find. The even-numbered problems will be graded carefully.

Section 2.3, Problems 1-4. These will not be graded carefully.

Draw a picture that explains why the difference quotient

\[
\frac{f(x + h) - f(x)}{h}
\]

gives the slope of a secant line to the curve \( y = f(x) \). Hint: your intuition may like this problem better if you think in terms of \( x_0 \) rather than \( x \). This problem will be graded carefully.